§5.3. L'Hôpital's Rule

Ex. $\lim_{x\to 0} \frac{e^x - 1}{x} = \frac{1-1}{0} = \frac{0}{0}$ indeterminate.

how do we calculate?

L'Hôpital's Rule: If a limit is indeterminate ($\frac{0}{0}$ or $\frac{\infty}{\infty}$), then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

$$\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} \frac{e^x}{1} = \lim_{x\to 0} e^x = e^0 = \boxed{1}$$

* We used this fact when we proved $\frac{d}{dx}[e^x] = e^x$

Ex. $\lim_{x\to 0} \frac{\ln(1+x^2)}{x^4} = \lim_{x\to 0} \frac{2x}{(1+x^2)4x^3} = \frac{2}{4} \lim_{x\to 0} \frac{1}{x^2(1+x^2)}$

$$= \frac{1}{2} \cdot \frac{1}{0} = \boxed{+\infty}$$

Ex. $\lim_{x\to\infty} \frac{3x^3}{5x^3+6} = \frac{3}{5}$

Ex. $\lim_{x \to \infty} \frac{\ln x}{x} = 0$

Ex. $\lim_{x \to \infty} \frac{e^x}{p(x)}$ for $p(x)$ any polynomial

WLOG $p(x) = x^n$

$n \gg 1$

$= \infty.$

Ex. $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{x^2}\right) = \lim_{x \to 0^+} \frac{x-1}{x^2} = -\infty$

Ex. $\lim_{x \to -4} \frac{x+4}{x^2-16}$

Ex. $\lim_{x \to -\infty} \frac{x}{\sqrt{4+x^2}} = 1$

5.1: First Derivatives & Graphs

What does the derivative tell us about the graph of a function?

if $f'(c) > 0$, then f is increasing at c.
if $f'(c) < 0$, then f is decreasing at c.
if $f'(c) = 0$, the f is constant at c.

The constant case is the most interesting.

it's also possible that $f'(c) = \text{DNE}$.

This can also be interesting.

Defn. The values of x in the domain of f with $f'(x) = 0$ or $f'(x) = \text{DNE}$ are called critical numbers or values of f.

* Critical ~~numbers~~ values for f are partition numbers for f'.

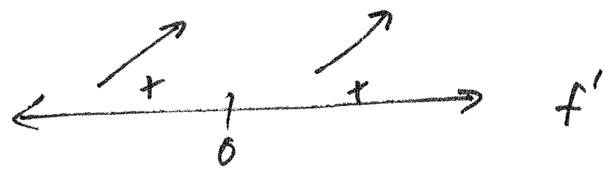
The derivative can only change signs at a critical values

Ex. $f(x) = 1+x^3$ Find the intervals where f is increasing, decreasing?

$f'(x) = 3x^2$

$f'(x) = 0 = 3x^2 \Rightarrow x = 0$

so $x=0$ is only CV.



$f'(-1) = 3(-1)^2 = 3 > 0$

$f'(1) = 3(1)^2 = 3 > 0$

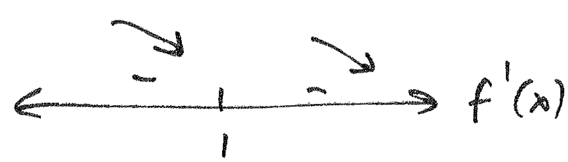
So f is inc on $(-\infty, 0) \cup (0, \infty)$

and f is dec no where.

Ex. $f(x) = (1-x)^{1/3}$

$f'(x) = \frac{1}{3}(1-x)^{-2/3}(-1) = \frac{-1}{3(1-x)^{2/3}}$

$f'(x)$ can never $= 0$, but $f'(x) = \text{DNE}$ if $x = 1$.



$$\left.\begin{array}{l} \text{if } x=0: \quad f'(0) = \frac{-1}{3} < 0 \\ \text{if } x=2 \quad f'(2) = \frac{-1}{3} < 0 \end{array}\right\} \quad \begin{array}{l} \text{dec: } (-\infty, 1) \cup (1, \infty) \\ \text{inc: neverrrrr} \end{array}$$

Ex. $f(x) = 8\ln(x) - x^2$

inc: $(0,2)$ dec: $(2,\infty)$

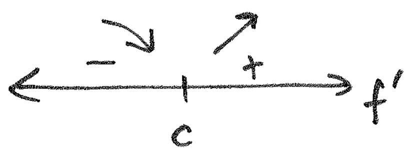
What does $f'(x) = 0$ mean?

$\leftarrow$ c is a CV of f.

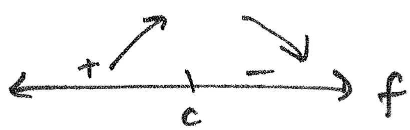
If $f'(c) = 0$ and f' changes signs at c then f has a local max or min at $x = c$.

First Derivative Test:

Let c be a CV of f. Construct a sign chart.



f(c) is a local min



f(c) is a local max

$+ \ +$ or $- \ -$ } not a local max or min.

(*) Ex. Find local extrema of $f(x) = x^3 - 6x^2 + 9x + 1$

If $p(x)$ is a polynomial then p can have at most n x-intercepts and at most n−1 local extrema.

where n is deg(p).

Ex. 84. $f(x) = \dfrac{x^2}{x+1}$

Find CN's, inc, dec, local extrema.

Ex. 96. Marginal Analysis. Show: profit will decrease over production intervals (a,b) where marginal rev is less than marg. cost.

5.2: Second Derivatives and Graphs