4.3. The Product and Quotient Rules:

We don't yet know how to take derivatives of stuff like this (products):

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = ?$$

This is not true: $\frac{d}{dx}\left[f(x)\cdot g(x)\right] \neq f'(x)\cdot g'(x)$.

Here's the product rule:

If f and g are both differentiable functions, then

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = \frac{d}{dx}\left[f(x)\right] g(x) + f(x)\frac{d}{dx}\left[g(x)\right]$$

OR

$$(fg)' = f'g + fg'$$

Exs. $f(x) = x^3(x+3x^2)$

$$f'(x) = 3x^2(x+3x^2) + x^3(1+6x)$$
$$= 3x^3 + 9x^4 + x^3 + 6x^4$$
$$= 4x^3 + 15x^4$$

Does this match the old way?

$f(x) = x^4 + 3x^5$

$f'(x) = 4x^3 + 15x^4$ Yup! :)

Ex. $f(x) = x^2 \ln(x)$

$f'(x) = 2x \ln(x) + x^2 \left(\frac{1}{x}\right)$

$= 2x \ln(x) + x$

Ex. $h(x) = 2^x 5^x$

$h'(x) = \ln 2 \; 2^x 5^x + \ln 5 \; 2^x 5^x$

$= (\ln 2 + \ln 5)\, 2^x 5^x$

$= \ln(10)\, 2^x 5^x$

Ex. $g(x) = e^x \log_5 x$

$g'(x) = e^x \log_5 x + e^x \frac{1}{\ln 5 \; x}$

Ex. $f(x) = x^9 e^x$

$f'(x) = 9x^8 e^x + x^9 e^x = x^8 (9e^x + xe^x)$

Ex. $f(x) = p(x) \cdot q(x)$ $\quad p(5) = 3 \quad p'(5) = 1$

$q(5) = 8 \quad q'(5) = 2$

Find $f'(5)$

$f'(5) = p'(5)\, q(5) + p(5)\, q'(5)$

$= 1 \cdot 8 + 3 \cdot 2 = 8 + 6 = 14$

Now for quotients:

The quotient rule:

If f and g are differentiable and $g \neq 0$, then

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

"low d-high minus high d-low all over low-low"

Ex. $f(x) = \dfrac{x^2 - 3x}{x}$

$$f'(x) = \frac{x(2x-3) - (x^2-3x)(1)}{x^2} = \frac{2x^2 - 3x - x^2 + 3x}{x^2}$$

$$= \frac{x^2}{x^2} = 1$$

Make sense?

$$f(x) = \frac{x^2 - 3x}{x} = \frac{x^2}{x} - \frac{3x}{x} = x - 3$$

$f'(x) = 1$ word.

Ex. $G(x) = \dfrac{x^2 - 2}{2x+1}$

$$G'(x) = \frac{(2x+1)(2x) - (x^2-2)(2)}{(2x+1)^2} = \frac{4x^2 + 2x - 2x^2 + 4}{(2x+1)^2}$$

$$= \frac{2x^2 + 2x + 4}{(2x+1)^2} = \frac{2(x^2+x+2)}{(2x+1)^2}$$

Ex. $y = \dfrac{\ln(x)}{x}$

$$y' = \frac{x\left(\frac{1}{x}\right) - \ln(x)}{x^2} = \frac{1 - \ln(x)}{x^2}$$

Ex. $f(x) = \dfrac{x}{x + \frac{c}{x}}$ for $c \in \mathbb{R}$

$$f'(x) = \frac{\left(x + \frac{c}{x}\right)(1) - x\left(1 - \frac{c}{x^2}\right)}{(x + c/x)^2} = \frac{x + \frac{c}{x} - x + \frac{c}{x}}{(x + c/x)^2}$$

$$= \frac{2c}{x(x + c/x)^2}$$

Ex. $y = \dfrac{\sqrt{x} - 1}{\sqrt{x} + 1}$

$$y' = \frac{(\sqrt{x}+1)\left(\frac{1}{2\sqrt{x}}\right) - (\sqrt{x}-1)\left(\frac{1}{2\sqrt{x}}\right)}{(\sqrt{x}+1)^2}$$

$$= \frac{\frac{1}{2} + \frac{1}{2\sqrt{x}} - \frac{1}{2} + \frac{1}{2\sqrt{x}}}{(\sqrt{x}+1)^2} = \frac{1}{\sqrt{x}(\sqrt{x}+1)^2}$$

Ex. Find equations of the tangent line to

$y = \dfrac{x^2 - 1}{x^2 + x + 1}$ at $(1, 0)$

$y = \frac{2}{3}(x - 1)$

$\boxed{y = \frac{2}{3}x - \frac{2}{3}}$

$$y' = \frac{(x^2+x+1)(2x) - (x^2-1)(2x+1)}{(x^2+x+1)^2}$$

$$y'(1) = \frac{(1^2+1+1)(2) - (1^2-1)(2(1)+1)}{(1^2+1+1)^2} = \frac{6 - 0}{9} = \frac{2}{3}$$

Ex. Find eqn's of tan lines to

$$y = \frac{x-1}{x+1}$$

parallel to $x - 2y = 2$

$$y' = \frac{(x+1) - (x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$$

slope of the line is $2y = x - 2$

$$y = \frac{1}{2}x - 1$$

$$m = 1/2$$

so solve: $\dfrac{2}{(x+1)^2} = \dfrac{1}{2}$

$$(x+1)^2 = 4$$

$$x^2 + 2x + 1 = 4$$

$$x^2 + 2x - 3 = 0$$

$$(x+3)(x-1) = 0$$

$$x = -3 \qquad x = 1$$

At $x = -3$:

$$y(-3) = \frac{-3-1}{-3+1} = \frac{-4}{-2} = 2$$

$$\cancel{y - \tfrac{1}{2} = \tfrac{1}{2}(x+3)}$$

$$\cancel{y = \tfrac{1}{2}x + \tfrac{3}{2} + \tfrac{1}{2}}$$

$$\boxed{\cancel{y = \tfrac{1}{2}x + 2}}$$

$$y - 2 = \frac{1}{2}(x+3)$$

$$y = \frac{1}{2}x + \frac{3}{2} + 2$$

$$\boxed{y = \frac{1}{2}x + \frac{5}{2}}$$

At $x = 1$:

$$y(1) = \frac{1-1}{1+1} = \frac{0}{2} = 0$$

$$y = \frac{1}{2}(x-1)$$

$$\boxed{y = \frac{1}{2}x - \frac{1}{2}}$$

Ex. Find $R'(0)$ for $R(x) = \dfrac{x - 3x^3 + 5x^5}{1 + 3x^3 + 6x^6 + 9x^9}$

Let $R(x) = \dfrac{f(x)}{g(x)}$

so $R'(x) = \dfrac{g(x) f'(x) - f(x) g'(x)}{g^2(x)}$

$f(0) = 0$ $\quad f'(x) = 1 - 9x^2 + 25x^4 \quad f'(0) = 1$

$g(0) = 1$ $\quad g'(x) = 9x^2 + 36x^5 + 81x^8 \quad g'(0) = 0$

so $R'(0) = \dfrac{g(0)\, f'(0) - f(0) g'(0)}{(g(0))^2} = \dfrac{1 \cdot 1 - 0 \cdot 0}{1^2} = \dfrac{1}{1} = \boxed{1}$

4.4. The Chain Rule

This is the most powerful derivative rule!

let $u(x)$ be a differentiable function and $f(x)$ also be differentiable. Then

$$[f(u)]' = f'(u) \cdot u'(x)$$

"Derivative of the outside times derivative of the inside"

Ex. $g(x) = (x+1)^2$

$f(u) = u^2$ $\quad$ $f'(u) = 2u$

$u(x) = x+1$ $\quad$ $u'(x) = 1$

so $g'(x) = 2(x+1) = 2x+2$

Old way: $g(x) = (x+1)^2 = x^2 + 2x + 1$

$g'(x) = 2x + 2$ ✓

Ex. $f(x) = (x+1)^{31}$

$f(u) = u^{31}$ $\quad$ $u(x) = x+1$

$f'(u) = 31\, u^{30}$ $\quad$ $u'(x) = 1$

$f'(x) = f'(u) \cdot u'(x)$

$= 31\,(x+1)^{30} \cdot 1 = 31\,(x+1)^{30}$

Ex. $f(x) = \ln(x^2+2x)$

$f(u) = \ln u \qquad u(x) = x^2+2x$

$f'(u) = \frac{1}{u} \qquad u'(x) = 2x+2$

$$f'(x) = f'(u) \cdot u'(x) = \frac{1}{u} \cdot 2x+2 = \frac{2x+2}{x^2+2x}$$

Ex. General Formula: $(\ln u)' = \frac{u'}{u}$

if $u = x$ then $u' = 1$ and this still works!

$y = \ln(\ln x)$

$$y' = \frac{u'}{u} = \frac{\frac{1}{x}}{\ln x} = \frac{1}{x \ln x}$$

Ex. Another general formula: $(e^u)' = e^u \cdot u'$

$y = e^{x^2+3x+2}$

$y' = e^u \cdot u' = e^{x^2+3x+2}(2x+3)$

Ex. General Power Rule: $(u^n)' = nu^{n-1} \cdot u'$

$y = (x^9+3x^2+2)^5$

$y' = 5(x^9+3x^2+2)^4(9x^8+6x)$

Ex. $y = \log_5(5^{x^2-1})$

Ex. $f(x) = (\ln x)^5$

$$f'(x) = 5(\ln x)^4 \cdot \frac{1}{x}$$

Ex. Find $\frac{d}{dw} \frac{1}{(w^3+4)^5} = \frac{d}{dw}(w^2+4)^{-5} = -5(w^2+4)^{-6}(2w)$

$$= \frac{-10w}{(w^2+4)^6}$$

Ex. Find the x-values where $f(x) = \sqrt{x^2+4x+5}$ has horizontal tangent line.

$$f'(x) = ?$$

General Formula: $(\sqrt{u})' = \frac{u'}{2\sqrt{u}}$

So $f'(x) = \frac{2x+4}{2\sqrt{x^2+4x+5}} = \frac{x+2}{\sqrt{x^2+4x+5}}$

$f'(x) = 0$ only when $x+2=0$

$$\boxed{x=-2}$$

check: $\sqrt{(-2)^2+4(-2)+5} = \sqrt{4-8+5} = \sqrt{1} = 1$ ok!

Tan line is: $\boxed{y=1}$