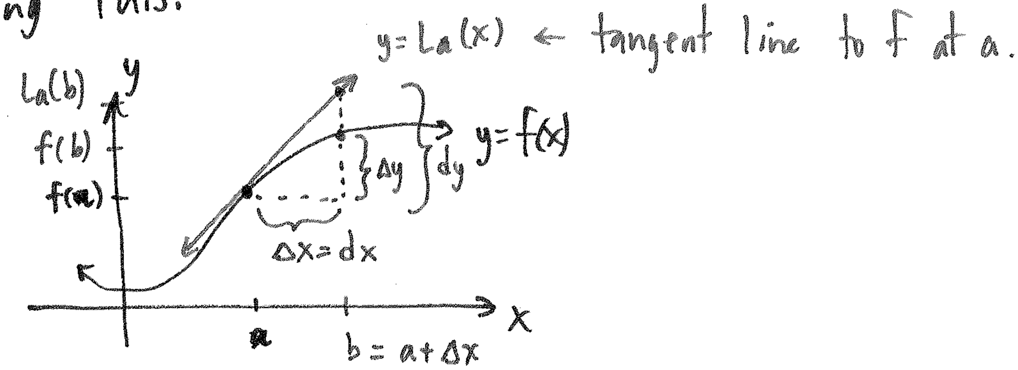
3.6. Differentials

Approximating functions near a point by the tangent line at the point. Differentials are the key to doing this.



$$f(a+dx) = f(a+\Delta x) = f(a) + \Delta y \approx f(a) + dy \quad , \text{so}$$

Approximation Formula: $\boxed{f(a+dx) \approx f(a) + dy} = L_a(x)$

derive the formula.

* dy is called the differential of $y = f(x)$.
* It represents the "vertical" change of the tangent line to the function $y = f(x)$.
* $y = L_a(x)$ is called the linearization of f at $x = a$.
  It is the tangent line to $y = f(x)$ at a.

Notice: the slope of $y = L_a(x)$ is $\frac{dy}{dx}$, the derivative of f at a. Now Leibniz's notation makes sense!

Question. How do we find dy?

Recall. $\frac{dy}{dx} = f'(x)$.

Here dx "is" a small number, but not zero, so we can move it to the other side of the $=$.

We get:

$$dy = f'(x)\,dx$$

If we want to ~~find dy~~ approximate f near a, we get

$$f(a+dx) \approx f(a) + dy = f(a) + f'(a)\,dx$$

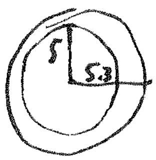
Ex. Approximate $\sqrt{9.5}$ using differentials.

$$f(x) = \sqrt{x}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$\begin{aligned} f(9.5) = f(9+0.5) &\approx f(9) + f'(9)(0.5) \\ &= \sqrt{9} + \frac{1}{2\sqrt{9}}(0.5) \\ &= 3 + \frac{1}{6}\cdot\frac{1}{2} \\ &= 3 + \frac{1}{12} \\ &= \boxed{\frac{37}{12}} \end{aligned}$$

Ex. An egg of a particular bird is nearly spherical. If the radius of the inside of the shell is 5 mm and the outside is 5.3 mm, what is the approx volume of the shell?



$$V = \frac{4}{3}\pi r^3$$

$$V_I = \frac{4}{3}\pi (5)^3 = \frac{4}{3}\pi (125) = \frac{500}{3}\pi$$

$$\frac{dV}{dr} = 4\pi r^2$$

so $dV = 4\pi r^2\, dr$

$$= 4\pi (5)^2 (.3)$$

$$= 4(25)\frac{3}{10}\pi = 30\pi$$

so $V_0 \approx V_I + dV = \frac{500}{3}\pi + 30\pi$

V of shell $= V_0 - V_I = \cancel{\frac{500}{3}}\pi + 30\pi - \cancel{\frac{500}{3}}\pi = \boxed{30\pi}$ mm

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Ex. Cost function: $C(x)$

Revenue function: $R(x)$

Profit Function: $P(x) = R(x) - C(x)$

Grills. $P(x) = \$20x - 0.02x^2 - 320 \quad 0 \le x \le 1{,}000$

a.) Find average profit for 40 grills produced/sold:

$$\text{Avg}: \overline{P}(x) = \frac{P(x)}{x} = \frac{P(40)}{40} = \frac{20(40) - 0.02(40)^2 - 320}{40} = 20 - 0.02(40) - 8$$

$$= 20 - 8 - .8$$

$$= \$11.2 \text{ / grill}$$

b.) Find marginal ~~cost~~ avg profit for 40 grills

$\bar{P}'(x) = \left(20 - 0.02x - \frac{320}{x}\right)' = -0.02 + \frac{320}{x^2}$

$\bar{P}'(40) = -0.02 + \frac{320}{1600} = -0.02 + 0.20 = \$0.18/\text{grill}$

c.) estimate the avg. profit for 41 grills:

$$\bar{P}(41) = \bar{P}(40+1) = \bar{P}(40) + \bar{P}'(40)(1)$$
$$= \$11.2 + \$0.18$$
$$\approx \boxed{\$11.38 \,/\text{grill}}$$

Ex. Estimate $(10.05)^2$.

Ex. Estimate $\sqrt[3]{126}$.

} F.T.I.S.
R.E.

4.1. Compound Interest Revisited

Continuously compound interest: $\boxed{A = Pe^{rt}}$

Ex. 10 yr. CD earns 4.15 % compounded cont.

A.) If \$10,000 is invested, how much is it worth after 10 yr?

$$A(10) = 10{,}000\, e^{0.0415(10)} = \boxed{\$15{,}143.71}$$

B.) How long will it take for the account to be worth \$18,000?

$$18{,}000 = 10{,}000\, e^{0.415t}$$

$$1.8 = e^{0.415t}$$

$$\ln(1.8) = 0.0415\, t$$

$$t = \frac{\ln(1.8)}{0.0415} = \boxed{14.16 \text{ yrs}}$$

Doubling Time: $A = 2P$ T = doubling time.

$$2\cancel{P} = \cancel{P}e^{rT}$$

$$2 = e^{rT}$$

$$\ln 2 = rT$$

$$\boxed{T = \frac{\ln 2}{r}}$$

Ex. How long will it take $1 to double if it's invested at 5% cont. int.?

$$T = \frac{\ln 2}{r} = \frac{\ln 2}{0.05} = \boxed{13.86 \text{ yrs}}$$

Radioactive Decay:

A cesium isotope has a half-life of 30 yrs. What is the cont. comp. rate of decay?

$$T = \text{half-life} = \frac{\ln 2}{r} =$$

(also)

$$30 = \frac{\ln 2}{r} \Rightarrow r = \frac{\ln 2}{30} = 0.0231$$

$$= 2.31\%$$

4.2. Derivatives of exp and logs.

Fact: $\boxed{\lim_{h\to 0} \frac{e^h - 1}{h} = 1}$

FTIS: verify this using Google, calculator, etc. (Excel?)

h	$\frac{e^h-1}{h}$
0.1	?
0.01	?
0.001	?
0.0001	?
$\downarrow$ 0	$\downarrow$ 1?

Now, calculate $f'(x)$ for $f(x)=e^x$ using limit def.

$$f(x+h) = e^{x+h} = e^x e^h$$

$$f(x) = e^x$$

$$f(x+h) - f(x) = e^x e^h - e^x = e^x(e^h - 1)$$

$$\frac{f(x+h)-f(x)}{h} = \frac{e^x(e^h-1)}{h}$$

$$f(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{e^x(e^h-1)}{h} = e^x \underbrace{\lim_{h\to 0} \frac{e^h-1}{h}}_{1}$$

$$= e^x \cdot 1 = e^x$$

so $\boxed{f'(x) = e^x}$

(*) $f(x) = e^x$ is the <u>only</u> function other than $f(x) = 0$ who has itself as its derivative!

<u>what about logs?</u>

let $f(x) = \ln(x)$

$f(x+h) = \ln(x+h)$

$f(x+h) - f(x) = \ln(x+h) - \ln(x) = \ln\left(\frac{x+h}{x}\right) = \ln\left(1 + \frac{h}{x}\right)$

$$\frac{f(x+h) - f(x)}{h} = \frac{1}{h}\ln\left(1 + \frac{h}{x}\right) = \frac{1}{x} \cdot \frac{x}{h} \ln\left(1 + \frac{h}{x}\right) = \frac{1}{x}\ln\left(1 + \frac{h}{x}\right)^{\frac{x}{h}}$$

$$f'(x) = \lim_{h \to 0} \frac{1}{x} \ln\left(1 + \frac{h}{x}\right)^{\frac{x}{h}}$$

let $y = \frac{x}{h}$, then $\lim_{h \to 0} y = \infty$

$$f'(x) = \frac{1}{x} \lim_{y \to \infty} \ln\left(1 + \frac{1}{y}\right)^y = \frac{1}{x} \ln\left[\lim_{y \to \infty}\left(1 + \frac{1}{y}\right)^y\right]$$

$$= \frac{1}{x}\ln(e) = \frac{1}{x} \cdot 1 = \frac{1}{x}$$

so, $\boxed{\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}}$

In general (we'll verify these later):

$$\frac{d}{dx}\left[a^x\right] = a^x \ln a \quad \text{and}$$

$$\frac{d}{dx}\left[\log_a x\right] = \frac{1}{x \ln a}.$$

F.T.I.S.: Verify that, using the formulae,

$$\frac{d}{dx}\left[e^x\right] = e^x \quad \text{and} \quad \frac{d}{dx}\left[\ln x\right] = \frac{1}{x}.$$

Ex. Find $\frac{dy}{dx}$ for $y = x^5 - 5^x$

$$\boxed{\frac{dy}{dx} = 5x^4 - \ln 5 \cdot 5^x}$$

Ex. Blood Pressure. x = weight in lbs
$p(x)$ = blood pressure in mm of mercury.

$$p(x) = 17.5(1 + \ln x) \qquad 10 \leq x \leq 100$$

Find inst. rate of change at $x = 10$, $x = 90$.

$$p'(x) = \frac{17.5}{x}$$

$$- \; p'(10) = \frac{17.5}{10} = 1.75 \text{ mm/lb}$$

$$- \; p'(90) = \frac{17.5}{90} = 0.19\overline{4} \text{ mm/lb}.$$

Ex. use differentials to approximate $2^{2.1}$

$f(x) = 2^x \qquad f'(x) = \ln 2 \cdot 2^x$

$dy = f'(a)\,dx = \ln 2 \cdot 2^2 (.1) = .4 \ln 2$

$f(2.1) \approx 2^2 + .4 \ln 2$ $\qquad$ in reality $2^{2.1} = 4.287$

$\approx 4 + .4 \ln 2$

$= 4.2773$