Recall: A derivative at a point is given by:

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

It represents the slope of the tangent line to $y = f(x)$ at $x = c$.

Ex. Find $f'(3)$ for $f(x) = x^3$.

$$f'(3) = 27$$

Derivatives as functions:

We can compute $f'(c)$ for any c in which $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ exists. This suggests that if we replace c by x, we can write $f'(x)$ as a function:

The derivative of f is given by

$$\frac{df}{dx}(x) = \frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

for all x that the limit exists.

Now we can use this to write a formula for f', then plug in a value for x.

Ex. Find $\frac{dy}{dx}$ for $y = x^3$.

$$\frac{dy}{dx} = 3x^2$$

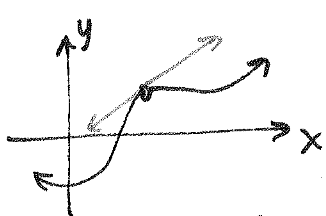
Ex. Find $f'(x)$ for $f(x) = \sqrt{x}$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

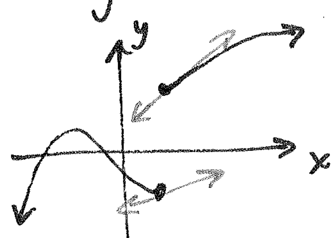
Notice: domain (f) : $x \geq 0$
domain (f') : $x > 0$ } We lose domain.

Why does this happen? i.e., what can go wrong?

discontinuities can go wrong:

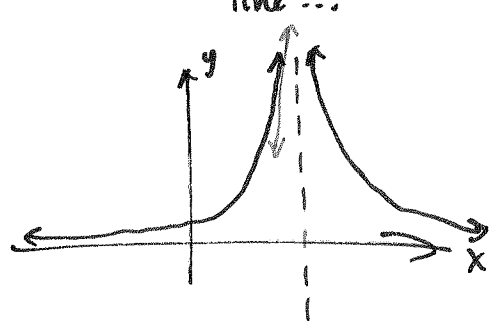


holes: tangent lines no longer "touch"



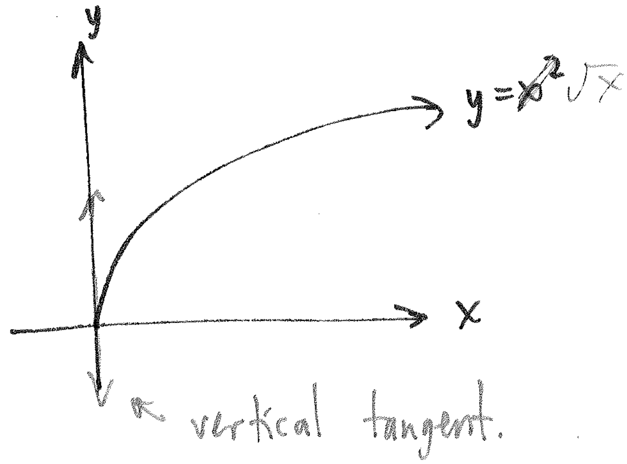
jumps: tangent lines may have the same slope, but they aren't the same line ...

Asymptotes:



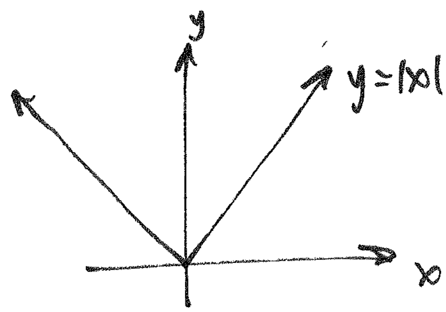
slope of secant lines $\longrightarrow \pm\infty$.

Also, ~~breaks~~ in $y = \cancel{x^2}\ \sqrt{x}$ case:



Moreover, continuous but with corners is bad:

Ex. $f(x) = |x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$



From the left:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{-x-h+x}{h} = -1$$

From the right:

$$f'(x) = +1$$

$+1 \neq -1$, so $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ does not exist at $x = 0$.

? Ex. Use limit rules and previous example to find $f'(x)$ for $f(x) = \frac{6}{x}$

recall: $\frac{d}{dx}\left[\frac{1}{x}\right] = \frac{-1}{x^2}$

$f(x+h) = \frac{6}{x+h}$ $f(x) = \frac{6}{x}$

$f(x+h) - f(x) = 6\left(\frac{1}{x+h} - \frac{1}{x}\right)$

so $f'(x) = 6\left(\frac{-1}{x^2}\right) = \boxed{\frac{-6}{x^2}}$

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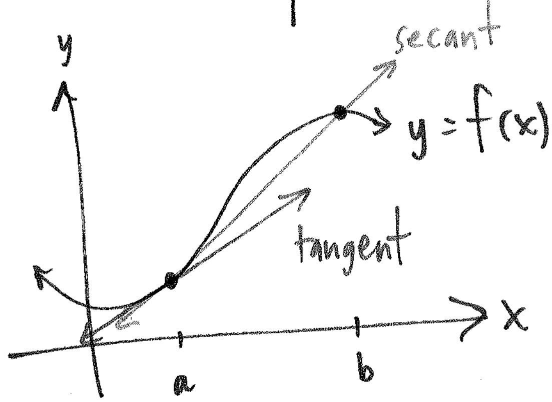
another interpretation of the derivative:



- The slope of a secant line represents the <u>average rate of change</u> of the function over an interval.
- The slope of a tangent line represents the <u>instantaneous rate of change</u> of the function at a point.

Ex. A company's total sales (in \$M) t months from now is given by

$$s(t) = 2\sqrt{t+6}$$

A. find $s'(t) = \frac{ds}{dt}$.

B. Find $s(10)$ and $s'(10)$. What do they mean?

C. Use part B to estimate the sales after 11 months and 12 months.

A. $\frac{ds}{dt} = \frac{1}{\sqrt{t+6}}$

B. $s(10) = 2\sqrt{10+6} = 2\sqrt{16} = 2(4) = \$8\text{ M}$

$s'(10) = \frac{1}{\sqrt{10+6}} = \frac{1}{\sqrt{16}} = \frac{1}{4} = \$\frac{1}{4}\text{ /month}$

C. $s(11) \approx s(10) + s'(10) = 8 + \frac{1}{4} = \$8.25\text{ M}$

$s(12) \approx s(10) + 2(s'(10)) = 8 + \frac{1}{2} = \$8.5\text{ M}$

// end of material for Exam 1

Section 3.5: Basic Derivative Rules

1. constant rule: $y = c$ for any # c

$$\Rightarrow \frac{dy}{dx} = 0 \quad \text{or} \quad \frac{d}{dx}[c] = 0$$

2. Power Rule: $y = x^n \Rightarrow \frac{dy}{dx} = n\,x^{n-1} \quad \forall n.$

3. Constant Multiple Rule: if $u(x)$ is a function with derivative $u'(x)$, and k is any #, then

$$\frac{d}{dx}[k\,u(x)] = k\,u'(x).$$

4. Sum/Difference Rules:

$$\frac{d}{dx}\left[u(x) \pm v(x)\right] = u'(x) \pm v'(x)$$

Ex. $f(x) = 3x^3 + x^2 + 2x - 1$

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}\left[3x^3 + x^2 + 2x - 1\right]$$

$$\overset{4.}{=} \frac{d}{dx}[3x^3] + \frac{d}{dx}[x^2] + \frac{d}{dx}[2x] - \frac{d}{dx}[1]$$

$$\overset{3.}{=} 3\frac{d}{dx}[x^3] + \frac{d}{dx}[x^2] + 2\frac{d}{dx}[x] - \frac{d}{dx}[1]$$

$$\overset{1./2.}{=} 3 \cdot 3x^2 + 2x + 2 \cdot 1 - 0$$

$$= \boxed{9x^2 + 2x + 2}$$

Ex. $y = \frac{1}{x} = x^{-1} \Rightarrow \frac{dy}{dx} = -1 \cdot x^{-2} = \boxed{\frac{-1}{x^2}}$

Ex. $f(x) = \sqrt{x} = x^{1/2} \Rightarrow f'(x) = \frac{1}{2} x^{-1/2} = \frac{1}{2} \frac{1}{x^{1/2}} = \boxed{\frac{1}{2\sqrt{x}}}$

Ex. $u = 3u^{2/3} - 5u^{1/3} \Rightarrow \frac{du}{dx} = 3 \cdot \frac{2}{3} u^{-1/3} - 5 \cdot \frac{1}{3} u^{-2/3}$

$$= \boxed{2u^{-1/3} - \frac{5}{3} u^{-2/3}}$$

Ex. Find the values of x where $y = f(x)$ has a horizontal tangent line.

$f(x) = 6x - x^2$

slope of tangent line $= f'(x)$

$f'(x) = 6 - 2x = 0$

$2x = 6$

$x = 3$

so $f'(3) = 0$ means tangent line to f at $x = 3$ is horizontal.

$\underline{\text{Ex}}$. $y = (2x-5)^2 \Rightarrow y' = 8x - 40$

$= 4x^2 - 40x + 25$

$\underline{\text{Ex}}$. $y = \dfrac{3x-4}{12x^2} = \dfrac{3x}{12x^2} - \dfrac{4}{12x^2} = \dfrac{1}{4}x^{-1} - \dfrac{1}{3}x^{-2}$

$$y' = -\frac{1}{4}x^{-2} + \frac{2}{3}x^{-3}$$

$$= -\frac{1}{4x^2} + \frac{2}{3x^3} = \frac{-3x}{12x^3} + \frac{8}{12x^3} = \boxed{\frac{8-3x}{12x^3}}$$

$\underline{\text{Ex}}$. Find an equation for the tangent line to

$y = 3x^2 + 2x + 1$ at $x = 2$

$$y(2) = 3(2)^2 + 2(2) + 1$$
$$= 12 + 4 + 1$$
$$= 17$$

$y' = 6x + 2$

$y'(2) = 6(2) + 2 = 14 =$ slope

$\underline{\text{recall}}$: pt-slope formula: $(y - y_1) = m(x - x_1)$

so,

$y - 17 = 14(x-2)$

$y = 14x - 28 + 17$

$\boxed{y = 14x - 11}$

End of material for Exam 1: No. back.

6/20/12