3.2. Exponential Equations

Ex. $2^x = 32$ solve for x.

$\cancel{2}^x = \cancel{2}^5$

$\boxed{x = 5}$

Ex. solve for x: $9^{x+2} = 3^x$

$(3^2)^{x+2} = 3^x$

$\cancel{3}^{2x+4} = \cancel{3}^x$

$2x + 4 = x$

$\boxed{x = -4}$

End: 4 Jun '12

3.3. Logarithms

11 Jun '12

Definition. $\log_a x = y$ means $a^y = x$ $a \neq 1,\ a > 0.$

Ex. $\log_2 8 = ?$

write $\log_2 8 = y \Rightarrow 2^y = 8$

$2^y = 2^3$ so $\log_2 8 = 3.$

$\boxed{y = 3}$

Logs are exponents.

$\log_{10} = \log =$ common log.

$$\begin{cases} \log_a 1 = 0 \\ \log_a a = 1 \end{cases}$$

3.4. Inverse Property

$y = \log_a x$ and $y = a^x$ are inverse functions to one another. So,

$$\log_a a^x = x,$$

and

$$a^{\log_a x} = x$$

They cancel each other out!

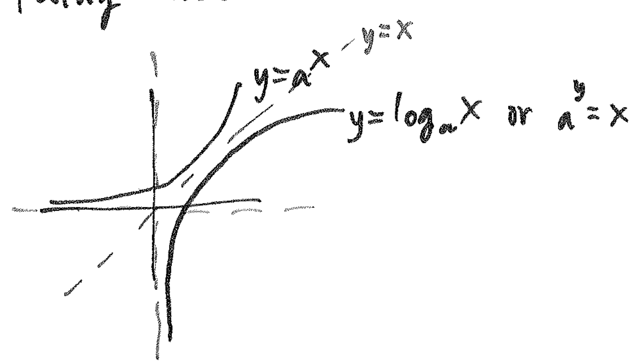
Ex. $\log_2 2^5 = 5$

$$\log_{10} 1000 = \log_{10} 10^3 = 3$$

This also tells us something about the domain and range of $\log_a$.

$D_{\log} = (0, \infty)$

$R_{\log} = (-\infty, \infty)$



3.5. The Natural Number, e

There is an amazing number that occurs so frequently in nature that it's called the natural number:

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x \approx 2.718281828459\ldots$$

Since $e>0$, $e\neq 1$ we can consider $y=e^x$.

This is a remarkable function from a calculus point of view, but we have to wait to truly appreciate that.

The inverse function to $y=e^x$ is $y=\log_e x$, the <u>natural logarithm</u>.

We write $y=\ln x$.

<u>Ex</u>. $\ln e^5 = 5$

$$\left.\begin{array}{l} \ln 2 = 0.69314 \\ \ln 11 = 2.39790 \end{array}\right\} \text{Calculator is necessary.}$$

$\ln 1 = 0.$

3.6. Log Rules $a>0, a\neq 1$, $M,N>0$, k any number.

<u>Product Rule</u>: $\log_a(MN) = \log_a M + \log_a N$

<u>Quotient Rule</u>: $\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$

<u>Power Rule</u>: $\log_a M^k = k\log_a M$.

<u>Ex</u>. $\log_9 27 = \log_9 \frac{81}{3} = \log_9 81 - \log_9 3 = 2 - \frac{1}{2} = \boxed{\frac{3}{2}}$

$$\begin{aligned} \log_a \frac{xy^2}{z^3} &= \log_a(xy^2) - \log_a(z^3) \\ &= \log_a x + \log_a y^2 - \log_a z^3 \\ &= \log_a x + 2\log_a y - 3\log_a z \end{aligned}$$

Ex. $\log w - \log z - 3\log x + 2\log y$

$= \log wy - \log zx^3$

$= \log\left(\frac{wy}{zx^3}\right).$

3.7. Exponential Growth and Decay

$$A = Pe^{rt}$$

$r > 0$: pop'n growth
continuously compounded interest

Ex. You invest \$1,850 in an account with 5% interest for 12 years. How much interest does the account earn.

$A = 1850\, e^{0.05(12)} = \$3370.92$

so Interest is $3370.92 - 1850 = \boxed{\$1520.92.}$

$r < 0$: decay

Ex.

3.8. Doubling Time $\quad A = Pe^{rt}$

T = time it takes P to double; i.e., $A = 2P$.

$$T = \frac{\ln 2}{r}$$

Ex. Derive this formula: $A = 2P$, so

$$2\cancel{P} = \cancel{P}e^{rT}$$

$$\ln 2 = \ln e^{rT}$$

$$\ln 2 = rT \cancel{\ln e}$$

$$\boxed{T = \frac{\ln 2}{r}}$$

Ex. You invest \$4,817.25 at 1.2% interest, compounded continuously. How long does it take for your investment to double?

$$T = \frac{\ln 2}{0.012} = \boxed{57.76 \text{ yrs.}}$$

What about 1.5% interest?

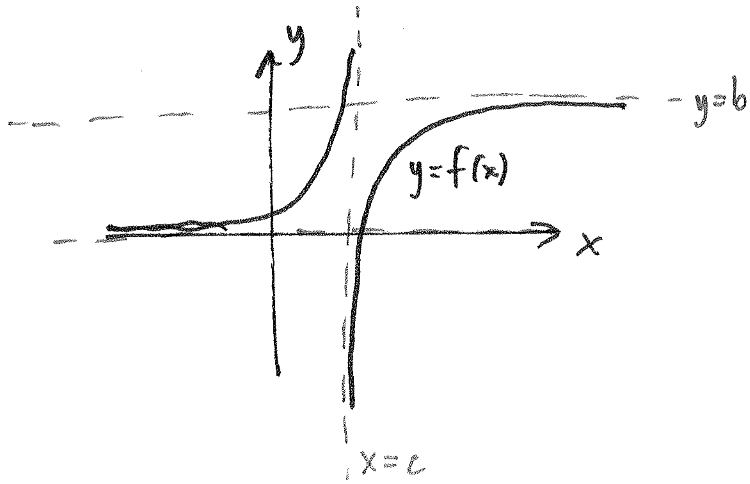
$$T = \frac{\ln 2}{0.015} = 46.21 \text{ yrs}$$

2%?

$$T = \frac{\ln 2}{2} = 34.66 \text{ yrs}$$

Back to § 3.2: — 11 Jun '12

Limits Involving Infinity



Vertical Asymptotes
Horizontal Asymptotes

some limits:

$\lim_{x \to c^-} f(x) = \text{DNE}$, but it's a special DNE

$\lim_{x \to c^-} f(x) = +\infty$

Similarly,

$\lim_{x \to c^+} f(x) = -\infty$

these DNE because ∞ is a "place", not a number.

$\lim_{x \to -\infty} f(x) = 0$

$\lim_{x \to \infty} f(x) = b$

these two exist!

These are all classified as limits involving infinity.

* There are rigorous definitions for all of these, but we'll skip them as the idea is most important.

Ex. $f(x) = \dfrac{2x}{x+2}$

find: $\lim\limits_{x\to -2} f(x)$, $\lim\limits_{x\to\infty} f(x)$, $\lim\limits_{x\to -\infty} f(x)$ if they exist.

$$\left.\begin{aligned} \lim_{x\to\infty} \frac{2x}{x+2} &= 2 \\ \lim_{x\to -\infty} \frac{2x}{x+2} &= 2 \end{aligned}\right\} \text{exist.}$$

$$\left.\begin{aligned} \lim_{x\to -2^-} \frac{2x}{x+2} &= \frac{2(-2^-)}{-2^- + 2} = \frac{-4}{0^-} = +\infty \\ \lim_{x\to -2^+} \frac{2x}{x+2} &= \frac{2(-2^+)}{-2^+ + 2} = \frac{-4}{0^+} = -\infty \end{aligned}\right\} \text{not } =, \text{ so } \lim_{x\to -2} \frac{2x}{x+2} = \text{DNE}.$$

3.2.1
Rules of calculation: Rational Functions

Let $f(x) = \dfrac{P(x)}{Q(x)}$, $Q(x) \neq 0$

$\deg P = m$
$\deg Q = n$

$\lim\limits_{x\to\pm\infty} f(x) =$

i) 0 if $m < n$

ii) DNE ($\pm\infty$) if $m > n$

iii) ratio of leading coefficients if $m = n$

More Exs involving ∞:

Ex. $f(x) = \dfrac{x+5}{x^2}$

Ex. $\dfrac{2x^2 - 5x + 2}{x^2 - x - 2} = g(x)$

Ex. $f(x) = \dfrac{x+1}{x^2-1}$

Ray Bradbury is dead. In his honor, we should all reread Fahrenheit 451.

That book is (in some sense) about technology poisoning society. In November, Bradbury reluctantly allowed F 451 to be made into an ebook. It was the only way a publisher would renew its license.

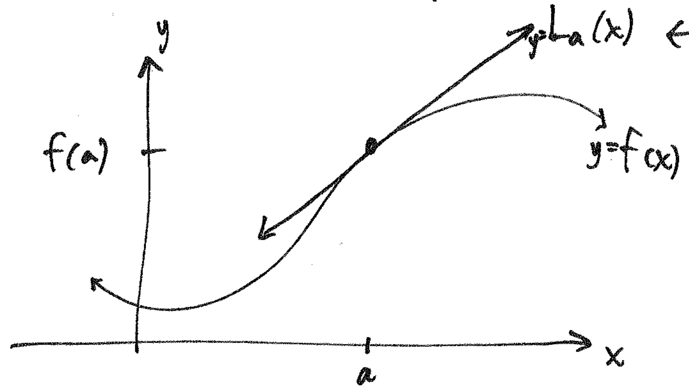
I can't help but think that Ray Bradbury, although he was 91, was killed by the Kindle (which ironically means "to set fire").

3.4 – The Derivative

— 11 Jun '12

3.4.1. The Definition:

(informal) the derivative of a function f at a point a is the slope of the tangent line to the function at the point.



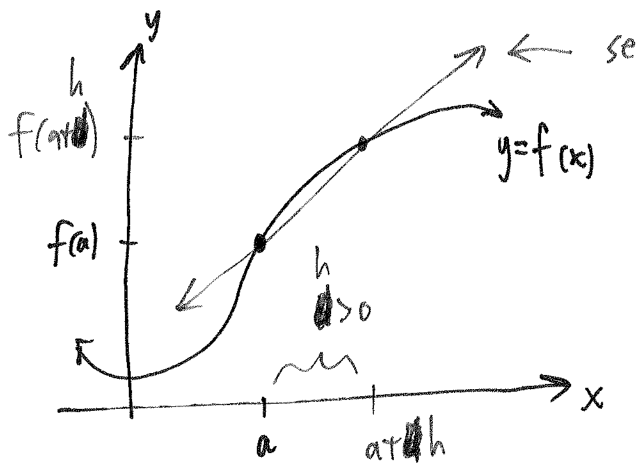
← tangent line to f at $x=a$.

the slope of $y=L_a(x)$ is the derivative of f at $x=a$.

3.4.2. How do we find it?

Start with a secant line (any secant line) and take the limit of the slope as the secant line approaches the tan line.

← secant line for $h>0$.



the slope of the secant line is given by:

$$m = \frac{f(a+h) - f(a)}{a+h - a} = \frac{f(a+h)-f(a)}{h}$$

The Difference Quotient!

As $h \to 0$ in the previous picture, the secant line becomes the tan line. More importantly, the slope of the secant lines approaches the slope of the tangent line (at $x=a$)!

So,

3.4.3. Limit Definition of Derivative:

The derivative of the function f at the point $x=a$ is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

The derivative does not exist if the limit does not exist.

Other notations: $\frac{df}{dx}(a)$, $\frac{d}{dx}f(a)$, $\left.\frac{dy}{dx}\right|_a$, $\dot{f}(a)$

$\frac{dy}{dx}$ is Leibniz's notation,

$\dot{f}$ is Newton's notation.

Leibniz's notation turns out to be much more useful and friendly mathematically despite the fact that Newton has claimed more of the popular fame for inventing calculus (they each invented it simultaneously and independently).

Ex. Find $f'(3)$ for $f(x)=x^3$.

$$f'(3) = \lim_{h\to 0} \frac{f(3+h)-f(3)}{h}$$

$$f(3+h) = (3+h)^3 = 3^3 + 3\cdot 3^2 h + 3\cdot 3h^2 + h^3$$
$$= 27 + 27h + 9h^2 + h^3$$

$$f(3+h) - f(3) = \cancel{27} + 27h + 9h^2 + h^3 - \cancel{27}$$
$$= 27h + 9h^2 + h^3$$
$$= h(27 + 9h + h^2)$$

$$\frac{f(3+h)-f(3)}{h} = \frac{\cancel{h}(27+9h+h^2)}{\cancel{h}} = 27 + 9h + h^2$$

$$f'(3) = \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0} 27 + 9h + h^2 = 27 + 9(0) + 0^2$$
$$= \boxed{27}$$

- We could do this computation for any a in the domain of f, so we may as well just do it for $y=f(x)$ instead of $f(a)$. We get:

<u>Defn</u>. The <u>derivative</u> of f is given by:

$$f'(x) = \frac{dy}{dx} = \frac{d}{dx}\left[f(x)\right] = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

* obviously this limit may not work for every choice of x, so the domain of f' could be different from f.

Ex. $y = f(x) = \frac{1}{x}$. Find $f'(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x+h) = \frac{1}{x+h}$$

$$f(x+h) - f(x) = \frac{1}{x+h} - \frac{1}{x} = \frac{x}{x(x+h)} - \frac{x+h}{x(x+h)} = \frac{-h}{x(x+h)}$$

$$\frac{f(x+h) - f(x)}{h} = \frac{\frac{-h}{x(x+h)}}{h} = \frac{-\cancel{h}}{x(x+h)} \cdot \frac{1}{\cancel{h}} = \frac{-1}{x(x+h)}$$

$$f'(x) = \frac{dy}{dx} = \lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x(x+0)} = \boxed{\frac{-1}{x^2}}$$

domain(f): $x \neq 0$, domain(f'): $x \neq 0$.

Ex. $f(x) = \sqrt{x}$, find $\frac{d}{dx}[f(x)]$

$$\left.\begin{array}{l} f(x+h) = \sqrt{x+h} \\ f(x) = \sqrt{x} \end{array}\right\} \quad \frac{f(x+h) - f(x)}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{\cancel{x} + \cancel{h} - \cancel{x}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})} = \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$f'(x) = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}}$$

Now, D_f: $x \geq 0$, but $D_{f'}$: $x > 0$ so the domain changed! sketch graph!