Chapter 3: Limits and the Derivative

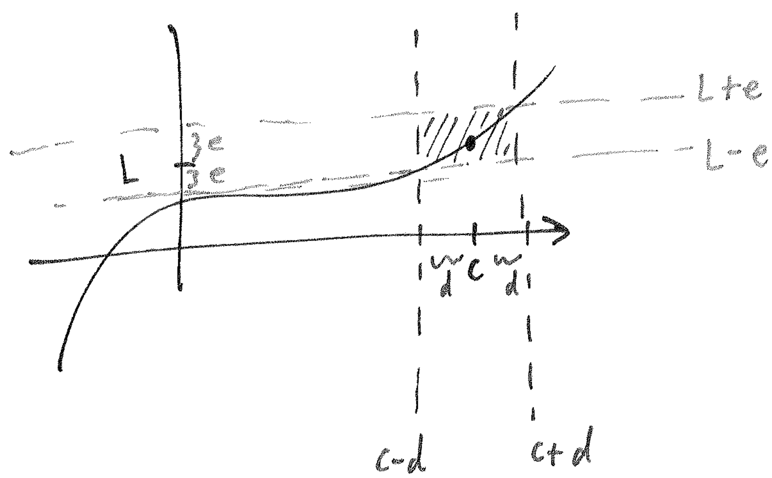
3.1. Introduction to Limits:

3.1.1. Rigorous definition: We say that the limit of f as x approaches c is L if and only if:

For every $e>0$, there exists a $d>0$ such that if $0<|x-c|<d$, then $|f(x)-L|<e$.

We write $\boxed{\lim_{x\to c} f(x) = L}$.

What does this mean? Let's draw a picture:



* x never actually gets to c!

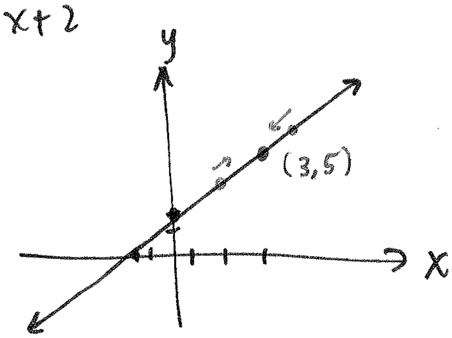
When $x \in (c-d, c+d)$, then $f(x) \in (L-e, L+e)$.

* As x gets arbitrarily close to c, the function's value, $f(x)$, gets arbitrarily to L.

3.1.2. Limits of graphs:

Pretend you can shrink yourself like Rick Moranis, and walk along the graph. When you get close to an ~~point~~ x-value, where are you on the y-axis?

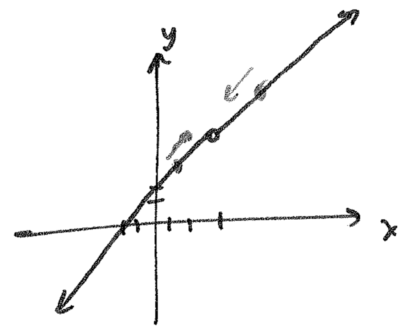
Ex. $f(x) = x+2$



Find $\lim_{x \to 3} f(x) = 5.$

Ex. $f(x) = x+2, \quad x \neq 3$

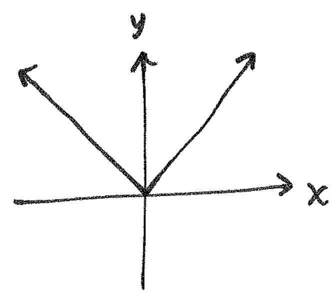
A hole in the graph above $x = 3$, but



$\lim_{x \to 3} f(x) = 5$ still!

* You need to approach the same height from above (from the right) and below (left).

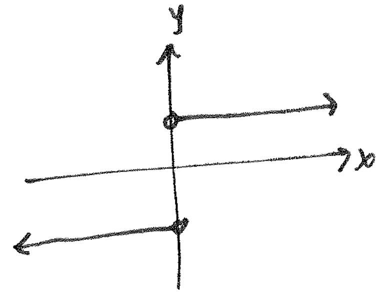
Ex. $f(x) = |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$



$\lim_{x \to 0} f(x) = 0.$

Ex. $f(x) = \frac{|x|}{x} = \begin{cases} -1 & x<0 \\ 1 & x>0 \end{cases}$

undef. if $x=0$



$\lim_{x\to 0} f(x) =$ DNE bc it's not the same from both sides.

This raises a question. Can we take "one-sided" limits? The answer is yes.

3.1.3 One-Sided Limits:

The limit as x approaches c from the left is written

$$\lim_{x\to c^-} f(x)$$

From the right:

$$\lim_{x\to c^+} f(x) .$$

Ex. $f(x) = \frac{|x|}{x}$

$$\lim_{x\to 0^-} f(x) = -1 ,$$

$$\lim_{x\to 0^+} f(x) = 1$$

So they each exist, but the limit itself does not.

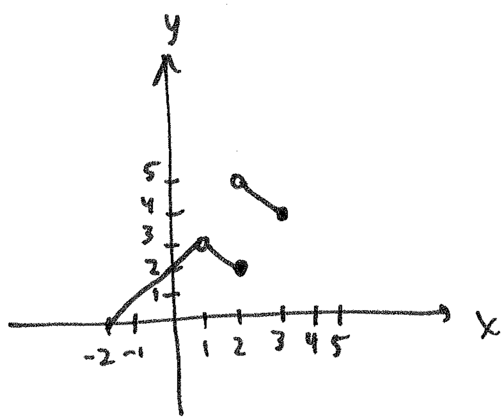
3.1.4. Existence of Limits:

Theorem. A limit exists if and only if the limits from the left and right exist, and are equal:

$$\lim_{x \to c} f(x) = L \iff \lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L.$$

SAVE this picture

Ex.



$\lim_{x \to 1} f(x) = 3$

$f(1) = \text{undef.}$

$\lim_{x \to 2^-} f(x) = 2$

$\lim_{x \to 2^+} f(x) = 5$

$\lim_{x \to 2} f(x) = \text{DNE}$

$f(2) = 2.$

Lots of weird stuff can happen.

We don't want to draw the graph of a function everytime we want to take a limit, so we need some rules for taking limits algebraically.

3.1.5. Rules of Calculation:

Theorem. Let f and g be two functions, and suppose

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M. \quad \text{Then}$$

1. $\lim_{x\to c} k = k$ for any constant k.

2. $\lim_{x\to c} x = c$

3/4. $\lim_{x\to c} [f(x) \pm g(x)] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x) = L \pm M$

5. $\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x) = kL$ for any constant k.

6. $\lim_{x\to c} [f(x)\, g(x)] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x) = L \cdot M$

7. $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \dfrac{L}{M}$ if $M \neq 0$.

8. $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)} = \sqrt[n]{L}$ if $L > 0$ for even n.

* Note: #6 doesn't imply this directly, but it's true:

$$\lim_{x\to c} [f(x)]^n = [\lim_{x\to c} f(x)]^n = L^n \quad \text{for any } n.$$

* Also note: you can replace $x \to c$ by $x \to c^+$ or $x \to c^-$ in any of these.

Ex. Find $\lim_{x\to 3}(x^2-4x)$

$$= \lim_{x\to 3} x^2 - \lim_{x\to 3} 4x$$

$$= \left[\lim_{x\to 3} x\right]^2 - 4\lim_{x\to 3} x$$

$$= 3^2 - 4(3)$$

$$= 9-12 = \boxed{-3}$$

Ex. Find $\lim_{x\to -1}\sqrt{2x^2+3}$

$$= \sqrt{\lim_{x\to -1}(2x^2+3)}$$

$$= \sqrt{2\left[\lim_{x\to -1} x\right]^2 + \lim_{x\to -1} 3}$$

$$= \sqrt{2(-1)^2+3}$$

$$= \boxed{\sqrt{5}}$$

Evaluation Theorem. If p is any polynomial function, then

$$\lim_{x\to c} p(x) = p(c)$$

If r is any rational function, then

$$\lim_{x\to c} r(x) = r(c) \quad \text{whenever the denominator} \neq 0.$$

Ex. $\lim_{x\to 2}(x^3-5x-1) = 2^3-5(2)-1 = 8-10-1 = \boxed{-3}$

Ex. $\lim_{x\to 4} \dfrac{2x}{3x+1} = \dfrac{2(4)}{3(4)+1} = \boxed{\dfrac{8}{13}}$

Cancelation Rule: If a rational function has common factors in the denominator and numerator, then you may cancel them inside of a limit ! only!

Ex. $\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} = \lim_{x \to 2} x+2 = 2+2 = \boxed{4}$

but $f(2) =$ undefined $\left(\frac{0}{0}\right)$.

Ex. $\lim_{x \to -1} \frac{x|x+1|}{x+1} = \lim_{x \to -1} x \cdot \lim_{x \to -1} \frac{|x+1|}{x+1} = -1 \cdot \text{DNE}$

$= \underline{\text{DNE}}$.

3.1.6. Limits of Quotients

If $\lim_{x \to c} f(x) = 0$ and $\lim_{x \to c} g(x) = 0$ then $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{0}{0} =$ ~~undef. or DNE.~~ indeterminate.

More specifically, it may still exist. We'll see how to deal with this in the future.

But, if $\lim_{x \to c} f(x) = L \neq 0$ and $\lim_{x \to c} g(x) = 0$, then the limit does not exist. It may "not exist" in a very special way. We will see this soon too.

3.1.7. Limits of Difference Quotients:

Recall (from Monday) that the difference quotient of a function is given by

$$DQ(f) = \frac{f(x+h) - f(x)}{h} \quad \text{for } h \neq 0.$$

Ex. For $f(x) = x^2$, find $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

First, find $DQ(f)$:

$$f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$$

$$f(x+h) - f(x) = \cancel{x^2} + 2xh + h^2 - \cancel{x^2}$$

$$= 2xh + h^2$$

$$\frac{f(x+h) - f(x)}{h} = \frac{2xh + h^2}{h} = \frac{\cancel{h}(2x+h)}{\cancel{h}} = 2x + h$$

Then, $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} 2x + h = 2x + 0 = \boxed{2x}$

Ex. For $f(x) = \sqrt{x}$, find $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$f(x+h) = \sqrt{x+h}$$

$$f(x+h) - f(x) = \sqrt{x+h} - \sqrt{x}$$

$$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{\sqrt{x} - \sqrt{x}}{0} = \frac{0}{0}$ $\stackrel{=}{\wedge}$ but we can fix it...

$$\lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})}{h}\frac{(\sqrt{x+h}+\sqrt{x})}{(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{\cancel{x}+h-\cancel{x}}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}}$$

FTIS. For $f(x)=\frac{1}{x}$, find $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

Jump to § 3.3: Continuity.

3.3.1: Definition: A function f is continuous at the point $x=c$ if and only if

1. $\lim_{x\to c} f(x)$ exists,
2. $f(c)$ exists, and
3. $\lim_{x\to c} f(x) = f(c)$.

A function is continuous on the open interval (a,b) if and only if it is continuous at every point in the interval.

* What does discontinuity look like?

1. holes
2. jumps
3. asymptotes

Ex: Continuity via the graph in 3.1.4.

Ex. Determine where the function is continuous:

$$m(x) = \frac{x+1}{(x-1)(x+4)}$$

The function is undefined at $x=1$, $x=-4$, so it is discontinuous there. It is continuous everywhere else:

$$x \neq 1,\ x \neq -4$$

Fact:

Polynomial functions are continuous everywhere.
Rational functions are continuous everywhere that they are defined; i.e., where the denominator $\neq 0$.

Ex. Where is this function discontinuous? What kind of discontinuities are they?

$$f(x) = \frac{x+1}{x^2-1} = \frac{x+1}{(x+1)(x-1)}$$

$x \neq 1$: asymptote

$x \neq -1$: hole.