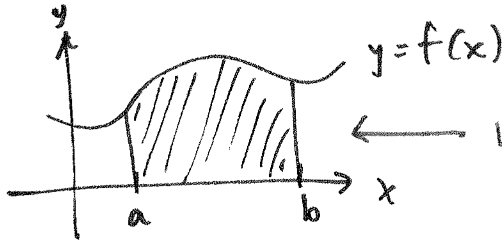
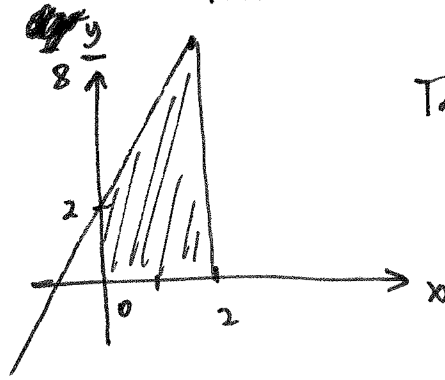
2. Definite Integrals: Area under a curve

Question:



$\longleftarrow$ what is the area of this region?

Ex. $y = 3x + 2$ from $x=0$ to $x=2$



Trapezoid: $A = \frac{1}{2} b\left(\frac{h_1+h_2}{2}\right)$

$b = \Delta x = 2$
$h_1 = f(0) = 2$
$h_2 = f(2) = 8$

Area $= \frac{1}{2}\cdot 2 \cdot \left(\frac{8+2}{2}\right) = \boxed{5}$

Ex. $y = \sqrt{9-x^2}$ from $x=0$ to $x=3$

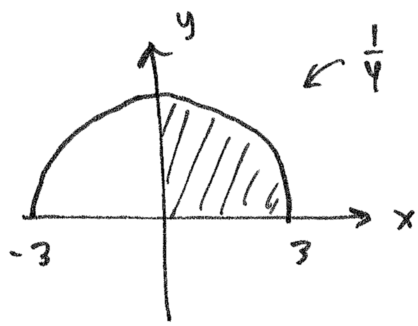
rearrange:

$y^2 = 9 - x^2$

$x^2 + y^2 = 9$

circle $c =$ origin
$r = 3$

top half only.



$\frac{1}{4}$ of a circle.

Area of circle $= \pi r^2$

so $A = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(3^2) = \boxed{\frac{9}{4}\pi}$

Ex. $y = 5$ from $x=-1$ to $x=3$

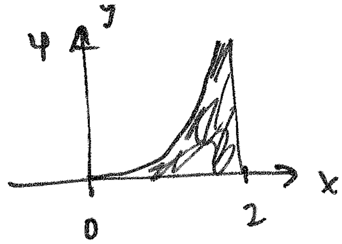
$A = 5(4) = \boxed{20}$

Ex. $y = 3x$ from $x = 0$ to $x = 5$

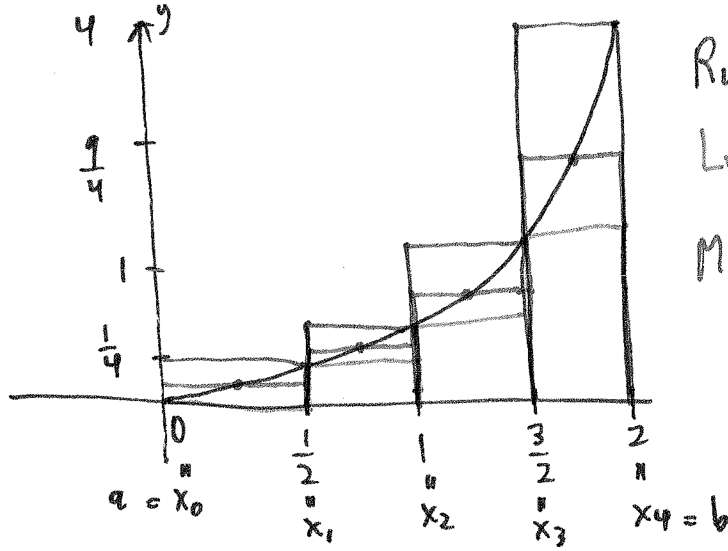
$$A = 5\left(\frac{15}{2}\right) = \boxed{\frac{75}{2}}$$

But what if the shape is too complicated?

i.e. $f(x) = x^2$ from $x = 0$ to $x = 2$.



We can approximate the area by "partitioning" the x interval, and using rectangles.



R_4 - over-estimate

L_4 - under-estimate

M_4 - "estimate" could be over or under.

Compute all of these and compare.

Finite Riemann Sums:

$$R_4 = \sum_{n=1}^{4} \frac{1}{2}\left(f(x_n)\right) \qquad M_4 = \sum_{n=1}^{4} \frac{1}{2}\left(f\left(\frac{x_n + x_{n-1}}{2}\right)\right)$$

$$L_4 = \sum_{n=1}^{4} \frac{1}{2}\left(f(x_{n-1})\right)$$

We can get better estimates by taking more partition points; i.e. more (narrower) rectangles.

It turns out that we can get the exact area by taking a limit. (go figure).

FIX NOTATION

$N = n$

$n = j$ or i

<u>Thm</u>. The area under the curve $y=f(x)$ on the interval $[a,b]$ is given by the <u>Riemann</u> <u>Sum</u>:

$$\boxed{\text{Area} = \lim_{N\to\infty} \sum_{n=1}^{N} f(x_n^*)\,\Delta x_n}$$

where x_n are partition points of $[a,b]$, $x_0=a$, $x_N=b$, $\Delta x_n = x_{n+1} - x_n$, and $x_n^* \in [x_{n-1}, x_n]$ is any point in the interval.

This is the "official" (most general) definition of the Riemann sum. Instead, we'll use this definition:

$$\boxed{\text{Area} = \lim_{N\to\infty} \sum_{n=1}^{N} f(x_n)\,\Delta x}$$

where x_n are "uniform" partition pts, $x_0=a$, $x_N=b$,

$\Delta x = \frac{b-a}{N}$, $x_n = a + \Delta x \cdot n$

Ex. Find exact area under $y=x^2$ from $x=0$ to $x=2$.

$$f(x)=x^2$$

$$a=0$$

$$\Delta x = \frac{b-a}{N} = \frac{2-0}{N} = \frac{2}{N}$$

$$x_n = a + \frac{2}{N}n = 0 + \frac{2}{N}\cdot n = \frac{2}{N}\cdot n$$

$$f(x_n) = \left(\frac{2}{N}n\right)^2 = \frac{4}{N^2}\cdot n^2$$

So Area $= \lim_{N\to\infty} \sum_{n=1}^{N} \left(\frac{4}{N^2}\cdot n^2\right)\left(\frac{2}{N}\right)$

$$= \lim_{N\to\infty} \left(\frac{8}{N^3}\right) \sum_{n=1}^{N} n^2$$

$$= \lim_{N\to\infty} \left(\frac{8}{N^3}\right)\left(\frac{N(N+1)(2N+1)}{6}\right)$$

$$= \lim_{N\to\infty} \left(\frac{8}{N^3}\right)\left(\frac{2N^3+\dots}{6}\right)$$

$$= \frac{8\cdot 2}{6} = \boxed{\frac{8}{3}} \ !$$

Some formulae:

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2} \leftarrow \text{R.E. Prove it.}$$

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{j=1}^{n} j^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Ex. Find area under $f(x)=x^3$ from $x=0$ to $x=1$.

$\dots$ Area $= \boxed{\frac{1}{4}}$.

R.E.: 1. Find R_6 for $y=x+x^2$ from $x=0$ to $x=3$.

2. Find exact area under #1. using R. sums.

3. Show $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$.