Ch. 6 - Not in order:

Integration & Antiderivatives

1. Antiderivatives and Differential Equations

A function F is an <u>antiderivative</u> <u>of</u> <u>f</u> iff $F' = f$.

<u>Ex</u>. $F(x) = \frac{x^3}{3}$ is an antiderivative of $f(x) = x^2$

b/c $F'(x) = \frac{3}{3}x^2 = x^2$.

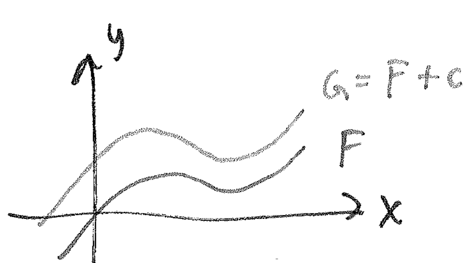
$F(x) = \frac{x^3}{3}$ is <u>not</u> the <u>only</u> antiderivative of f.

$G(x) = \frac{x^3}{3} + 9$ $\qquad G'(x) = x^2 = f(x)$.

In fact, $F(x) = \frac{x^3}{3} + c$ for any $c \in \mathbb{R}$ is an antideriv. of f.

<u>Thm</u>. Let F and G be diff'able functions on an interval (a,b), and $F' = G'$ on (a,b). Then $F = G + c$ for some $c \in \mathbb{R}$.

i.e., F and G differ by <u>at most</u> a constant (a vertical translation of the graph)



Ex. Find all antiderivatives of $f(x) = x$.

since $\frac{d}{dx}\left(\frac{x^2}{2}\right) = x$, therefore $\boxed{F(x) = \frac{x^2}{2} + c.}$

Note: $F(x) = \frac{x^2}{2} + c$ is a family of functions. There is a different function for every $c \in \mathbb{R}$.

Ex. Find all antiderivatives of $g(x) = x^3 + x^2 + 1$

$\frac{d}{dx}\left(\frac{x^4}{4}\right) = x^3 \qquad \frac{d}{dx}\left(\frac{x^3}{3}\right) = x^2 \qquad \frac{d}{dx}(x) = 1$

so $\boxed{G(x) = \frac{x^4}{4} + \frac{x^3}{3} + x + c}$

Antiderivatives of polynomials:

if $f(x) = x^n$, then $\boxed{F(x) = \frac{x^{n+1}}{n+1} + c}$ is its antideriv. for all $n \neq -1$.

(*) Literally backward differentiation. Add a power, then divide by the new power.

Ex. $f(x) = 2x^2 + 3x + 5$

$F(x) = \frac{2}{3}x^3 + \frac{3}{2}x^2 + 5x + c$

Ex. $f(x) = x^9 + 13x^7 + 9x$

$F(x) = \frac{1}{10}x^{10} + \frac{13}{8}x^8 + \frac{9}{2}x^2 + c$

Ex. $g(x) = -7x^3 + 3x^2 + 2x - 2$

$G(x) = \frac{-7}{4}x^4 + \frac{3}{3}x^3 + \frac{2}{2}x^2 - 2x + c$

$= \boxed{-\frac{7}{4}x^4 + x^3 + x^2 - 2x + c}$

Antiderivative of e^x:

if $f(x) = e^x$, then $F(x) = e^x + c$ is its antiderivative.

Ex. $f(x) = x + 3e^x$

$F(x) = \frac{1}{2}x^2 + 3e^x + c$

Antiderivative of $\frac{1}{x}$:

If $f(x) = \frac{1}{x}$ then $F(x) = \begin{cases} \ln x + c & \text{if } x > 0 \\ \ln(-x) + c & \text{if } x < 0 \end{cases}$

or $F(x) = \ln|x| + c$

is its antiderivative

Ex. $f(x) = e^x + \frac{2}{x} - \frac{4}{x^2} = e^x + \frac{2}{x} - 4x^{-2}$

$F(x) = e^x + 2\ln|x| + \frac{4}{x} + c$

Ex. $f(x) = -\frac{3}{x^5} + \frac{1}{x^2} - \frac{1}{x} = -3x^{-5} + x^{-2} - \frac{1}{x}$

$$F(x) = \frac{3}{4}x^{-4} - x^{-1} - \ln|x| + c$$

$$= \frac{3}{4x^4} - \frac{1}{x} - \ln|x| + c$$

- As we can see, antiderivatives "skip" multiplied constants, and distribute over sums and differences.

In summary:

$f(x)$	$F(x)$
k	$kx + c$
x^n, $n \neq -1$	$\frac{1}{n+1}x^{n+1} + c$
$\frac{1}{x}$	$\ln\|x\| + c$
e^x	$e^x + c$

Differential Equations:

We know the instantaneous rate of change of a function, $f'(x)$, and want to find the function itself, $f(x)$.

Ex. The marginal cost of producing x units is given by

$$C'(x) = 0.3x^2 + 2x$$

And the fixed cost is $C(0) = \$2{,}000$.

"initial condition"

Find the cost function $C(x)$, and the cost of producing 20 units.

If $C'(x) = 0.3x^2 + 2x$, then $C(x) = \frac{0.3}{3}x^3 + \frac{2}{2}x + c$

$= 0.1x^3 + x + c$

Use the I.C. to find "c".

$C(0) = 0.1(0)^3 + 0 + c = 2000$

so $c = 2000$ and

$\boxed{C(x) = 0.1x^3 + x + 2000}$

Plug in $x = 20$, $C(20) = 0.1(20)^3 + 20 + 2000$

$= 0.1(8000) + 20 + 2000$

$\boxed{= \$2{,}100}$

Ex. 88. The rate of change of monthly sales of a new football game is

$S'(t) = 500\, t^{1/4}$, $S(0) = 0$,

where t is # of months since release.

When will monthly sales reach 20,000 games?

$S(t) = \frac{4}{5} 500\, t^{5/4} + c = 400\, t^{5/4} + c$ $\quad S(0) = 0 = 0 + c$ so $c = 0$.

$S(t) = 400\, t^{5/4} = 20000$ $\quad t = 50^{4/5} \approx \boxed{22.87 \text{ mos.}}$

$t^{5/4} = 50$

Ex. 94. The area A of a healing wound changes at a rate of approx.

$$\frac{dA}{dt} = -4t^{-3} \qquad 1 \le t \le 10$$

t is time in days, and $A(1) = 2\,cm^2$. What will area be in 10 days?

$$\frac{dA}{dt} = -4t^{-3} \longrightarrow A(t) = \frac{-4}{-2}t^{-2} + c$$

$$= \frac{2}{t^2} + c$$

$$A(1) = \frac{2}{1^2} + c = 2\ cm^2 \quad \Rightarrow c = 0$$

so $A(t) = \frac{2}{t^2}$

then $A(10) = \frac{2}{10^2} = \frac{2}{100} = 0.02\ cm^2$

Recommed Exercises:

Find the antiderivatives of:

1. $f(x) = 10x^{3/2}$

Also, pp. 360-1: # 89, 93, 95

2. $u(v) = \frac{4}{v} + \frac{v}{4}$
3. $y = x - e^x$
4. $\frac{dx}{dt} = 5t^{-1} + 1$
5. $f(x) = 3\sqrt{x} + \frac{2}{\sqrt{x}}$