Precalculus Mathematics:

Lecture Notes

By

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Introduction
To
Precalculus Mathematics

Dear Students,

Any regular calculus course at the university level will require that students know certain basic mathematics stuffs as a prerequisite. These basic stuffs will not be taught in those calculus courses. Instead, a calculus course will straightaway get into its nitty-gritty subject matter. The instructors of calculus courses will take it that students already know these basic stuffs prior to signing up for their calculus classes.

However, not all students would have studied these basic mathematics stuffs when they sign up for a calculus course. Hence, the purpose of this course is to prepare students with these basic stuffs for the more rigorous calculus classes that they may be required to take or perhaps just for the fun of taking a calculus course. The contents in this book are chosen based on what we know to be stuffs that calculus instructors assume students in their classes would already have studied and what we believe will be helpful to you when you embark on your actual calculus course.

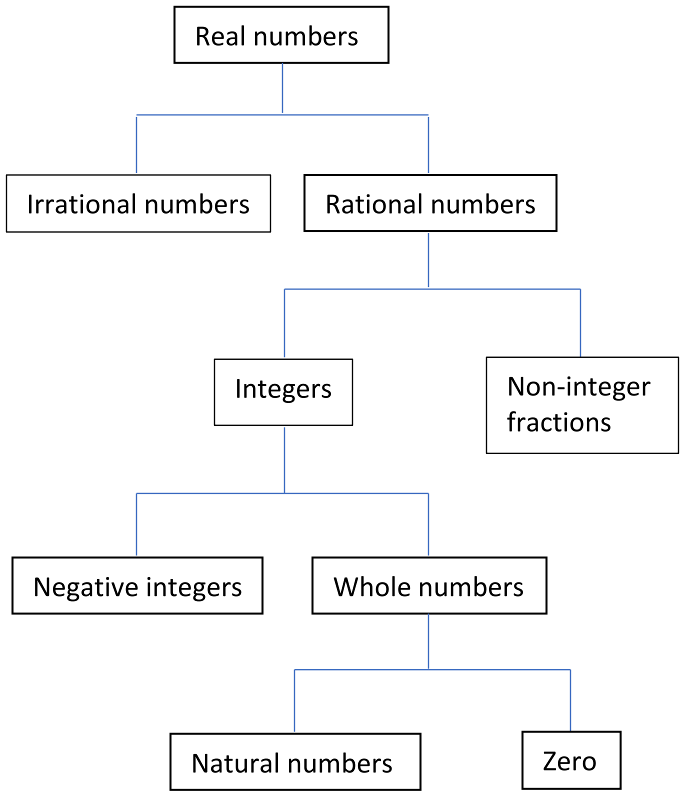
Here is an analogy of why this precalculus course is important especially to students who did not take enough mathematics classes while in school. Taking a calculus course is like taking classes on learning how to run. However, in order to be able to learn how to run, first of all, you need to know how to walk. Not knowing enough basic mathematics is not knowing how to walk properly where calculus is concerned. Therefore, this precalculus mathematics course aims to teach you how to walk and be able to confidently take a course on learning how to run, or to confidently take a calculus course.

Sincerely,
Dr Husam Khader

LIST OF MODULES

1.1 Real Numbers

Real numbers are numbers that exist. Below is a chart detailing the relationship between the types of real numbers.



1. **Natural numbers** (symbol **N**)

$$N = \{1, 2, 3, 4, \cdots\}$$

2. **Whole numbers** (symbol **W**)

$$W = \{0, 1, 2, 3, 4, \cdots\}$$

3. **Integers** (symbol **Z**)

$$Z = \{\cdots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \cdots\}$$

4. **Rational numbers** (symbol **Q**)

These are numbers that can be written as exact fractions. For example,

$$0.5 = \frac{1}{2}, \ \frac{1}{3} = 0.333333\cdots = 0.\overline{3}$$

Rational numbers are ratios of integers, i.e. $\frac{m}{n}$ where *m* and *n* are integers and $n \neq 0$.

Important: Division by zero is not allowed, because

1. $\frac{m}{0}$ is undefined if $m \neq 0$

And 2. $\frac{0}{0}$ is indeterminate. Its limit value depends on the situation it is defined.

All rational numbers can be written as decimals with either

1. Repeating patterns of decimal digits

Examples: $\frac{1}{3} = 0.33333333 \cdots = 0.\overline{3}$

$\frac{22}{7} = 3.142857142857142857 \cdots = 3.\overline{142857}$

Or 2. Exact terminating decimal representations

Examples: $\frac{1}{2} = 0.500000 \ldots = 0.5$

$\frac{3}{8} = 0.3750000 \ldots = 0.375$

Conversely, all decimal numbers with either repeating representations of decimal digits or exact terminating decimal representations can be written as exact fractions.

Note: All integers are rational numbers. If *a* is an integer, then $a = \frac{a}{1}$.

Examples: $5 = \frac{5}{1}$

$\frac{123}{1} = 123$

Converting a decimal representation of a rational number into its equivalent fraction

Example: If the decimal number terminates. For example, let us write 0.875 in its equivalent fraction.

We can write that

$$0.875 = \frac{0.875}{1}$$

We see that there are three digits to the last non-zero digit in the decimals. Therefore, as we move the decimal point rightwards in the numerator, we will write three zeroes after the 1 in the denominator to obtain

$$0.875 = \frac{0.875}{1} = \frac{875}{1000}$$

Another example: If the decimal digits are repeating. For example, let us write $3.652652652\cdots = 3.\overline{652}$ in its equivalent fraction.

This process is a bit complicated but nonetheless interesting. The first step is to let this number be represented by *x*. In this example, we let

$$x = 3.652652652\cdots$$

We see that there are 3 digits in each repeating group. So let us multiply both sides by 1,000, i.e. 1 followed by three zeroes. We will have

$$1000x = 3652.652652652\cdots$$

If we subtract as follows,

$$1000x = 3652.652652652\cdots$$
$$-x = -3.652652652\cdots$$

$\Rightarrow$

$$999x = 3649$$

$\Rightarrow$

$$x = \frac{3649}{999} = 3\frac{652}{999}$$

And Another example: Let us write $0.571428571428571428\cdots = 0.\overline{571428}$ in its equivalent fraction.

Let $x = 0.571428571428571428\cdots$.

We see there are six digits in each repeating group, so we multiply both sides by 1,000,000 to obtain

$$1{,}000{,}000x = 571428.571428571428571428\cdots$$

And we subtract

$$1{,}000{,}000x = 571428.571428571428571428\cdots$$
$$-x = -0.571428571428571428\cdots$$

$\Rightarrow$

$$999{,}999x = 571428$$

$\Rightarrow$

$$x = \frac{571428}{999999} = \frac{4}{7}$$

5. **Irrational numbers** (symbol **I**)

These are numbers that cannot be written as exact fractions. For example,

$$\pi = 3.141592653589793\cdots$$

$$e = 2.718281828459\cdots$$

$$\sqrt{3} = 1.73205080756887\cdots$$

Irrational numbers have decimal digits that

1. do not follow a repetitive pattern.

And 2. continue infinitely.

Properties of Real Numbers

1. Commutative Properties

i. $a + b = b + a$

Example: $2 + 3 = 3 + 2 = 5$

ii. $ab = ba$

Example: $3 \times 5 = 5 \times 3 = 15$

2. Associative Properties

iii. $(a + b) + c = a + (b + c)$

Example:
$$\begin{aligned} &(2+3)+5 \\ =\ & \quad 5 \quad +5 = 10 \end{aligned}$$

OR

$$\begin{aligned} &2+(3+5) \\ =\ &2+ \quad 8 \quad = 10 \end{aligned}$$

iv. $(ab)c = a(bc)$

Example:
$$\begin{aligned} &(2\times 3)\times 5 \\ =\ & \quad 6 \quad \times 5 = 30 \end{aligned}$$

OR

$$\begin{aligned} &2\times(3\times 5) \\ =\ &2\times \quad 15 \quad = 30 \end{aligned}$$

3. Distributive Property

v. $a(b+c)$

Example:

$$\begin{aligned} & 2\times(3+5) \\ &= 2\times 3 + 2\times 5 \\ &= \quad 6 \quad + \quad 10 \ = 16 \end{aligned}$$

OR

$$\begin{aligned} & 2\times(3+5) \\ &= 2\times \quad 8 \quad = 16 \end{aligned}$$

Addition and Subtraction

Note: A subtraction is the addition of a negative number.

Example:

$$\begin{aligned} & 3-5 \\ &= 3+(-5) = -2 \end{aligned}$$

Multiplication and Division

Note: A division is the multiplication of the reciprocal of a divisor

Example: $5 \div 2 = \frac{5}{2}$

$= 5 \times \frac{1}{2} = \frac{5}{2}$

By the same token, we extend this to fractions:

Example: $\frac{a}{b} \div \frac{c}{d}$

$= \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$ We multiply by the reciprocal of the divisor

Properties of Negatives

1. $(-1)a = -a$

2. $-(-a) = a$

3. $(-a)b = -ab = a(-b)$

4. $(-a)(-b) = ab$

5. $-(a+b) = (-a)+(-b) = -a-b$

6. $-(a-b) = -a + b = b - a$

Properties of Fractions

1. $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$, provided $b \neq 0$ and $d \neq 0$.

2. $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$, provided $b \neq 0$, $c \neq 0$ and $d \neq 0$.

3. $\frac{a}{b} \pm \frac{c}{b} = \frac{a \pm c}{b}$, provided $b \neq 0$. Add or subtract numerators with common denominators.

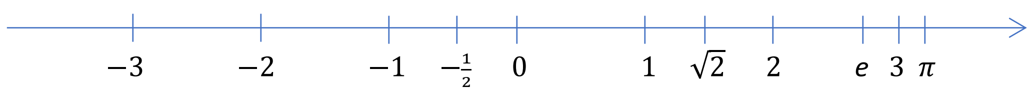
4. $\frac{a}{b} \pm \frac{c}{d} = \frac{ad}{bd} \pm \frac{bc}{bd} = \frac{ad \pm bc}{bd}$, provided $b \neq 0$ and $d \neq 0$.

$\frac{a}{xy} \pm \frac{b}{xz} \pm \frac{c}{yz} = \frac{az}{xyz} \pm \frac{by}{xyz} \pm \frac{cx}{xyz} = \frac{az \pm by \pm cx}{xyz}$, provided $x \neq 0$, $y \neq 0$ and $z \neq 0$.

We create common denominators in order to be able to add or subtract the numerators.

5. $\frac{a}{b} = \frac{ac}{bc}$, provided $b \neq 0$ and $c \neq 0$.

Real Number Line



The arrow on the real number line indicates the positive direction. We never draw an arrow head in the negative direction.

One-to-one correspondence:

1. Every real number corresponds to one point on the real number line.

2. Every point on the real number line corresponds to one real number.

Every number on the real number line is called the **coordinate** of that point.

The number zero on the real number line is called the **origin**. It is a neutral number, i.e. it is neither positive nor negative. When we say non-negative numbers, we include both zero as well as positive numbers. Likewise, when we say non-positive numbers, we include both zero and negative numbers. When we say positive numbers, we do not include zero. And when we say negative numbers, we do not include zero.

Order of Real Numbers

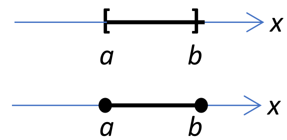
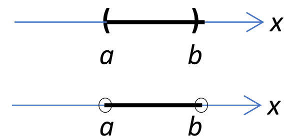
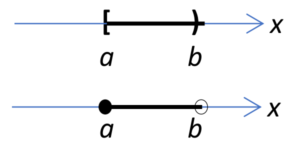
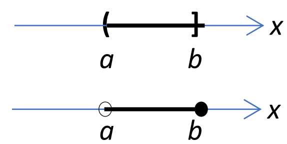
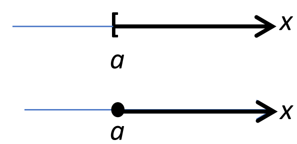
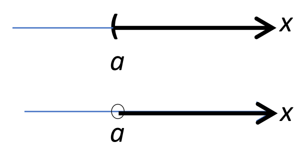
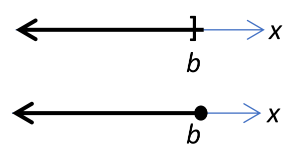
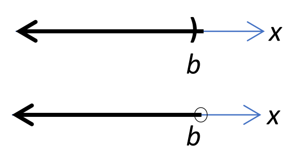
If a and b are two real numbers and that $a < b$ (or $b > a$). The position of a is to the left of b (or the position of b is to the right of a) on the real number line. In summary, on the number line,

Number at left < Number at right

or

Number at right > Number at left

Intervals on the Real Number Line

Interval Notation	Inequality Notation	Graph
$[a, b]$	$\{x \;/\; a \leq x \leq b\}$	 Or
(a, b)	$\{x \;/\; a < x < b\}$	 Or
$[a, b)$	$\{x \;/\; a \leq x < b\}$	 Or
$(a, b]$	$\{x \;/\; a < x \leq b\}$	 Or
$[a, \infty)$	$\{x \;/\; x \geq a\}$	 Or
(a, ∞)	$\{x \;/\; x > a\}$	 Or
$(-\infty, b]$	$\{x \;/\; x \leq b\}$	 Or
$(-\infty, b)$	$\{x \;/\; x < b\}$	 Or
$(-\infty, \infty)$	$\{x \;/\; -\infty < x < \infty\}$	

Unions and Intervals of Sets and Intervals

Given two sets, *S* and *T*, then

1. Union of *S* and *T*

$$S \cup T$$

is the set of all elements in *S* or in *T* or in both *S* and *T*.

2. Intersection of *S* and *T*

$$S \cap T$$

is the set of all elements common only to both *S* and *T*.

Examples:

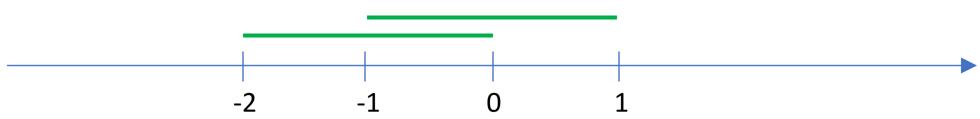
1. Find the indicated set if $A = \{1, 2, 3, 4, 5, 6, 7\}$ and $B = \{2, 4, 6, 8\}$

(a) $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$

(b) $A \cap B = \{2, 4, 6\}$

2. $(-2,0) \cup (-1,1) = (-2,1)$

3. $(-2,0) \cap (-1,1) = (-1,0)$



Definition of Absolute Value

If *a* is a real number, then the absolute value of *a* is

$$|a| = \begin{cases} a, & if\ a \geq 0 \\ -a, & if\ a < 0 \end{cases}$$

Examples:

1. $|5| = 5$

In this case, $a = 5$, which is a non-negative number. Therefore, $|5| = 5$.

2. $|-5| = 5$

In this case, $a = -5$, which is a negative number. Therefore, $|-5| = -(-5) = 5$.

3. $-|-6| = -6$

4. What is $\frac{|x|}{x}$?

$$\frac{|x|}{x} = \begin{cases} \frac{x}{x}, & if\ x > 0 \\ Indeterminate, & if\ x = 0 \\ \frac{-x}{x}, & if\ x < 0 \end{cases}$$

$$\Rightarrow \quad \frac{|x|}{x} = \begin{cases} 1, & if\ x > 0 \\ Indeterminate, & if\ x = 0 \\ -1, & if\ x < 0 \end{cases}$$

Properties of Absolute Values

If *a* and *b* are two real numbers, then

1. $|a| \geq 0$

2. $|-a| = |a|$

3. $|ab| = |a||b|$

4. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$, provided $b \neq 0$

Distance between two points on the real number line

If *a* and *b* are two real numbers, then the distance between *a* and *b* is

$$|b - a| = |a - b|$$

Example: Find the distance between -25 and 13.

$$|-25 - 13| = |-38| = 38$$

Or $\quad |13 - (-25)| = |13 + 25| = |38| = 38$

SUMMARY

Algebraic Expression

Terminologies:

Terms: The parts in an expression separated by the *addition* (+) or *subtraction* (-) operators.

Factors: The parts in a term that multiply or divide each other.

BEDMAS Rule

When we simplify or solve a mathematics expression, the following operators follow the BEDMAS hierarchy.

B – Brackets
E – Exponent
D – Division
M – Multiplication
A – Addition
S – Subtraction

Note: Addition and Subtraction have the same hierarchy, i.e. either one could be performed first.

Examples:

1. $$\begin{aligned} & 2 \times 3 + 4 \div 2 \\ = & \ \ 6 \ \ + \ \ 2 \\ = & \ \ \ \ \ 8 \end{aligned}$$

2. $$\begin{aligned} & (2+3) \times (4-2) \\ = & \ \ 5 \ \ \times \ \ 2 \\ = & \ \ \ \ \ 10 \end{aligned}$$

3. $$\begin{aligned} & 5 - 3^2 \\ = & \ 5 - 9 \\ = & \ -4 \end{aligned}$$

4. $$\begin{aligned} & 5 + (-3)^2 \\ = & \ 5 + \ \ 9 \\ = & \ 14 \end{aligned}$$

5. $$\begin{aligned} & 5 + 8 \div 2 \times 3 \\ = & \ 5 + 4 \times 3 \\ = & \ 5 + 12 = 17 \end{aligned}$$

Rules of Algebra

What is it called?	Property
Subtraction	$a - b = a + (-b)$
Division	$\frac{a}{b} = a\left(\frac{1}{b}\right)$, provided $b \neq 0$
Commutative property of addition	$a + b = b + a$
Commutative property of multiplication	$ab = ba$
Associative property of addition	$(a + b) + c = a + (b + c)$
Associative property of multiplication	$(ab)c = a(bc)$
Distributive property	$a(b + c) = ab + ac$ $(a + b)(c + d) = ac + ad + bc + bd$
What is it called?	**Property**
Additive identity property	$a + 0 = a$
Multiplicative identity property	$a \times 1 = a$
Additive inverse property	$a + (-a) = a - a = 0$
Multiplicative inverse property	$a \times \frac{1}{a} = \frac{a}{a} = 1$, provided $a \neq 0$

Some rules of multiplication:

1. $(+ve) \times (+ve) = +ve$

2. $(+ve) \times (-ve) = -ve$

3. $(-ve) \times (+ve) = -ve$

4. $(-ve) \times (-ve) = +ve$

Properties of negation and equality

1. $(-1)a = -a$

2. $-(-a) = a$

3. $(-a)b = -ab = a(-b)$

4. $(-a)(-b) = ab$

5. $-(a + b) = (-a) + (-b) = -a - b$

6. If $a = b$, then $a \pm c = b \pm c$. Conversely, if $a \pm c = b \pm c$, then $a = b$.

7. If $a = b$, then $ac = bc$. However, the converse is not necessarily true. If $ac = bc$, then $a = b$ only if $c \neq 0$.

Properties of zero:

1. $a \pm 0 = a$

2. $a \times 0 = 0$

3. $\frac{0}{a} = 0$, provided $a \neq 0$

4. $\frac{a}{0}$ is undefined if $a \neq 0$

5. $\frac{0}{0}$ is indeterminate.

6. **Zero-Factor property:** If $ab = 0$, then $a = 0$ or $b = 0$.

Properties and Operations of fractions:

1. $\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc$

2. $-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$

3. $\frac{-a}{-b} = \frac{a}{b}$

4. $\frac{a}{b} = \frac{ac}{bc}$, provided $c \neq 0$

5. $\frac{a}{b} \pm \frac{c}{b} = \frac{a \pm c}{b}$, provided $b \neq 0$. Add or subtract with like denominators.

6. $\frac{a}{b} \pm \frac{c}{d} = \frac{ad}{bd} \pm \frac{bc}{bd} = \frac{ad \pm bc}{bd}$, provided $b \neq 0$ and $d \neq 0$. We create like denominators in order to be able to add or subtract.

7. $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$, provided $b \neq 0$ and $d \neq 0$.

8. $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$

General Rule of Thumb when treating both sides of an equation: Be fair to both sides.

1. If we add a value to one side of an equation, we have to add the same value to the other side.

Example: $a = b$
$\Rightarrow$ $a + c = b + c$

2. If we subtract a value from one side of an equation, we have to subtract the same value from the other side.

Example: $a = b$

$\Rightarrow$ $a - d = b - d$

3. If we multiply a value to one side of an equation, we have to multiply the same value to the other side.

Example: $a = b$

$\Rightarrow$ $ac = bc$

4. If we divide one side of an equation by a non-zero value, we have to divide the other side by the same value.

Example: $a = b$

$\Rightarrow$ $\frac{a}{c} = \frac{b}{c}$, provided $c \neq 0$

General Rule of Thumb when moving a term from one side of the equation to the other.

When we move a term, we change its sign, i.e. positive becomes negative and negative becomes positive.

1. $a + b = c$

$\Rightarrow$ $a = c - b$

Proof: $a + b = c$

$\Rightarrow$ $a + b - b = c - b$

$\Rightarrow$ $a = c - b$

2. $a - b = c$

$\Rightarrow$ $a = c + b$

Proof: $a - b = c$

$\Rightarrow$ $a - b + b = c + b$

$\Rightarrow$ $a = c + b$

General Rule of Thumb when moving a factor from one side of the equation to the other.

When we move a factor from one side of the equation to the other, if it was originally in the numerator, we move it to the denominator at the other side. And if it was originally in the denominator, we move it to the numerator at the other side.

1. $\frac{a}{b} = \frac{c}{d}$

$\Rightarrow$ $\frac{1}{b} = \frac{c}{ad}$

Proof: $\frac{a}{b} = \frac{c}{d}$

$\Rightarrow \quad \frac{a}{b} \times \frac{1}{a} = \frac{c}{d} \times \frac{1}{a}$

$\Rightarrow \quad \frac{1}{b} = \frac{c}{ad}$

2. $\quad \frac{a}{b} = \frac{c}{d}$

$\Rightarrow \quad a = \frac{bc}{d}$

Proof: $\frac{a}{b} = \frac{c}{d}$

$\Rightarrow \quad \frac{a}{b} \times b = \frac{c}{d} \times b$

$\Rightarrow \quad a = \frac{bc}{d}$

1.2 Exponents and Radicals

Exponential Notation

If *a* is a real number and *n* is a positive integer, then we define

$$\underbrace{a \times a \times a \times \cdots \times a}_{n \text{ factors of } a} = a^n$$

We say that *a* is the **base**
and that *n* is the **exponent**.

Examples:

1. $3 \times 3 = 3^2 = 9$

2. $5 \times 5 \times 5 = 5^3 = 125$

3. $(-2) \times (-2) \times (-2) = (-2)^3 = -8$

4. $(-3) \times (-3) \times (-3) \times (-3) = (-3)^4 = 81$

Properties of Exponents

1. $a^m a^n = a^{m+n}$

Proof: $a^m a^n = a^m \times a^n$

$$= \underbrace{(a \times a \times a \times a \times \cdots \times a)}_{m \text{ factors of } a} \times \underbrace{(a \times a \times a \times \cdots \times a)}_{n \text{ factors of } a}$$

$$= \underbrace{a \times a \times a \times a \times a \times \cdots \times a}_{(m+n) \text{ factors of } a}$$

$$= a^{m+n}$$

2. $\dfrac{a^m}{a^n} = a^{m-n}$, provided $a \neq 0$.

Proof: $\dfrac{a^m}{a^n} = \dfrac{\overbrace{\cancel{a}\times\cancel{a}\times\cancel{a}\times\cancel{a}\times\cdots\times a}^{m \text{ factors of } a}}{\underbrace{\cancel{a}\times\cancel{a}\times\cancel{a}\times\cdots\times\cancel{a}}_{n \text{ factors of } a}}$

$$= \underbrace{a \times a \times \cdots \times a}_{(m-n) \text{ factors of } a}$$

$$= a^{m-n}$$

Zero Exponent

3. $a^0 = 1$, provided $a \neq 0$.

Proof: From the above property 2, we can have

$$\frac{a^n}{a^n} = a^{n-n}$$

$$= a^0$$

Also, $\dfrac{a^n}{a^n} = \dfrac{\overbrace{\cancel{a}\times\cancel{a}\times\cancel{a}\times\cdots\times\cancel{a}}^{n \text{ factors of } a}}{\underbrace{\cancel{a}\times\cancel{a}\times\cancel{a}\times\cdots\times\cancel{a}}_{n \text{ factors of } a}}$

$$= 1$$

Combining the above two results, we have

$\dfrac{a^n}{a^n} = a^0 = 1$, provided $a \neq 0$.

Note: 0^0 is indeterminate.

Negative Exponent

4. $a^{-n} = \dfrac{1}{a^n}$, provided $a \neq 0$.

Proof: $a^{-n} = a^{0-n}$

$= \dfrac{a^0}{a^n}$ From property 2.

$= \dfrac{1}{a^n}$ From property 3.

Note: Similarly, we can show that

$$a^n = \frac{1}{a^{-n}}$$

More Laws of Exponents

5. $(ab)^m = a^m b^m$

Proof: $(ab)^m = \underbrace{ab \times ab \times ab \times \cdots \times ab}_{m \text{ factors of } ab}$

$= \underbrace{a \times a \times a \times \cdots \times a}_{m \text{ factors of } a} \times \underbrace{b \times b \times b \times \cdots \times b}_{m \text{ factors of } b}$

$= \quad a^m \quad \times \quad b^m$

$= a^m b^m$

6. $(a^m)^n = a^{mn}$

Proof: $(a^m)^n = a^m \times a^m \times a^m \times \cdots \times a^m$ ← n factors of a^m

$= \underbrace{\underbrace{(a \times a \times \cdots \times a)}_{m \text{ factors of } a} \times \underbrace{(a \times a \times \cdots \times a)}_{m \text{ factors of } a} \times \underbrace{(a \times a \times \cdots \times a)}_{m \text{ factors of } a} \times \cdots \times \underbrace{(a \times a \times \cdots \times a)}_{m \text{ factors of } a}}_{n \text{ factors of } m \text{ factors of } a}$

$= a \times a \times a \times a \times a \times \cdots \times a$ ← mn factors of a

$= a^{mn}$

7. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

Proof: $\left(\frac{a}{b}\right)^m = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \times \cdots \times \frac{a}{b}$ ← m factors of $\frac{a}{b}$

$= \frac{a \times a \times a \times \cdots \times a}{b \times b \times b \times \cdots \times b}$ ← m factors of a / m factors of b

$= \frac{a^m}{b^m}$

Corollary of these laws

[8] $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

[9] $\frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n}$

More examples:

[1] $\left(\frac{1}{3}\right)^4 (-3)^2 = 3^{-4}3^2 = 3^{-4+2} = 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

[2] $\left(\frac{5}{3}\right)^0 2^{-1} = (1)\left(\frac{1}{2}\right) = \frac{1}{2}$

[3] $\left(\frac{1}{4}\right)^{-2} = 4^2 = 16$

Scientific Notation

This is an efficient way of writing and calculating

1. very large numbers positively or negatively, and
2. numbers that are very close to zero.

Format of a number written in scientific notation:

$$\pm a \times 10^n$$

Where $1 \leq a < 10$
And *n* is an integer.

We need to **remember** that

$1 \times 10 = 10$
$1 \times 10^2 = 100$
$1 \times 10^3 = 1\ 000$
$1 \times 10^4 = 10\ 000$
$\cdots$
$1 \times 10^{10} = 10\ 000\ 000\ 000$

And

$1 \times 10^{-1} = 0.1$
$1 \times 10^{-2} = 0.01$
$1 \times 10^{-3} = 0.001$
$1 \times 10^{-4} = 0.000\ 1$
$\cdots$
$1 \times 10^{-10} = 0.000\ 000\ 000\ 1$

Examples:

1. The speed of light is approximately 299,792,458 metres per second or about 300,000,000 ms^{-1}.

   Or in scientific notation, it is written as approximately $3 \times 10^8\ ms^{-1}$.

2. The Avogadro constant is approximately $6.022142 \times 10^{23}\ mol^{-1}$.

3. The Gravitational constant is approximately $6.6738 \times 10^{-11}\ m^3kg^{-1}s^{-2}$.

Evaluation using scientific notation

We use the properties of exponents discussed earlier.

Example: Evaluate $\frac{2{,}400{,}000{,}000 \times 0.000\ 004\ 5}{0.000\ 03 \times 1{,}500}$

Answer: $\frac{2{,}400{,}000{,}000 \times 0.000\ 004\ 5}{0.000\ 03 \times 1{,}500} = \frac{(2.4\times10^{9}) \times (4.5\times10^{-6})}{(3\times10^{-5}) \times (1.5\times10^{3})}$

$$= \frac{2.4\times4.5}{3\times1.5} \times \frac{10^{9}\times10^{-6}}{10^{-5}\times10^{3}}$$

$$= \frac{2.4\times4.5}{4.5} \times \frac{10^{9-6}}{10^{-5+3}}$$

$$= 2.4 \times \frac{10^{3}}{10^{-2}}$$

$$= 2.4 \times 10^{3-(-2)}$$

$$= 2.4 \times 10^{3+2} = 2.4 \times 10^{5} = 240{,}000$$

Another example:

$$(7.2 \times 10^{-9})(1.806 \times 10^{-12}) = (7.2)(1.806) \times (10^{-9})(10^{-12})$$

$$= 13.0032 \times 10^{-9-12}$$

$$= 13.0032 \times 10^{-21}$$

Radicals (Anything that has a root sign is called a radical)

We know that $b^2 = b \times b$. If we say that $b^2 = b \times b = a$, then the inverse of it is

$$b = \sqrt[2]{a}$$

Similarly, if we have $c^3 = c \times c \times c = a$, then the inverse of it is

$$c = \sqrt[3]{a}$$

Example:

We know that $3^2 = 9$. The inverse of it is

$$\sqrt[2]{9} = 3$$

Note: When we write $\sqrt[2]{x}$, we can omit writing the digit 2. So $\sqrt[2]{x} = \sqrt{x}$. But be careful, this omission is only allowed for square roots, not for any other root.

n^{th} root of a number

Let *a* and *b* be two real numbers such that $a = b^n$. Then, *b* is the n^{th} root of *a* and is denoted by the radical symbol

$$\sqrt[n]{a} = b$$

We call *n* the index of the radical
And *a* the radicand.

Another example: What is the cube root of 64?

Answer: We know that $64 = 4^3$. Therefore,

$$\sqrt[3]{64} = 4$$

Sign notation of an even root

We always define the sign of an even root by the sign we affix to the front of its radical.

For example, $\sqrt{4} = 2$

And $-\sqrt{4} = -2$

It is wrong to write $\sqrt{4} = -2$

Another example,

$$\sqrt[4]{81} = 3$$

And $-\sqrt[4]{81} = -3$

Generalisations of $\sqrt[n]{a}$

Real number *a*	**Natural number *n***	**Roots of *a***	**Examples**
$a > 0$	*n* is even	$\sqrt[n]{a}$, $-\sqrt[n]{a}$	$\sqrt[4]{81} = 3$, $-\sqrt[4]{81} = -3$
$a > 0$ or $a < 0$	*n* is odd	$\sqrt[n]{a}$	$\sqrt[3]{8} = 2$, $\sqrt[3]{-8} = -2$
$a < 0$	*n* is even	$\sqrt[n]{a}$ does not exist	$\sqrt{-4}$ does not exist
$a = 0$	*n* is even or *n* is odd	$\sqrt[n]{0} = 0$	

Properties of radicals

1. $\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

2. $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$

3. $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$, provided $b \neq 0$

4. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

5. $\left(\sqrt[n]{a}\right)^n = a$

6. $\sqrt[n]{a^n} = \begin{cases} |a|, & if\ n\ is\ even \\ a, & if\ n\ is\ odd \end{cases}$

Examples:

1. $\left(\sqrt{3}\right)^2 = 3$

2. $\left(\sqrt{-3}\right)^2 = -3$. It is one of those strange quirks that happen in mathematics.

3. $\sqrt{(-3)^2} = \sqrt{9} = 3$

4. $\sqrt[3]{27} = 3$

5. $\sqrt[3]{-27} = -3$

6. $-\sqrt[3]{-27} = -(-3) = 3$

Rational Exponents

We recall that in the definition of roots, if $a = b^n$, then $\sqrt[n]{a} = b$. Now, suppose we take both sides to the power of *n*. We have

$$\sqrt[n]{a} = b$$

$$\Rightarrow \quad \left(\sqrt[n]{a}\right)^n = b^n$$

$$\Rightarrow \quad \left(\sqrt[n]{a}\right)^n = a \text{ , we started with } a = b^n$$

Now, we take both sides to the power of $\frac{1}{n}$,

$$\left(\left(\sqrt[n]{a}\right)^n\right)^{\frac{1}{n}} = a^{\frac{1}{n}}$$

$\Rightarrow \quad \left(\sqrt[n]{a}\right)^{n\times\frac{1}{n}} = a^{\frac{1}{n}}$, using the property $(r^s)^t = r^{st}$

$\Rightarrow \quad \sqrt[n]{a} = a^{\frac{1}{n}}$

This is a very important result. From here, we can obtain two more results:

$$\sqrt[n]{a} = a^{\frac{1}{n}}$$

$\left(\sqrt[n]{a}\right)^m = \left(a^{\frac{1}{n}}\right)^m$, taking the left and right sides to the power of *m*

$\Rightarrow \quad \left(\sqrt[n]{a}\right)^m = a^{\frac{m}{n}}$, once again using the property $(r^s)^t = r^{st}$

The other result:

$$\sqrt[n]{a} = a^{\frac{1}{n}}$$

$\Rightarrow \quad \sqrt[n]{a^m} = (a^m)^{\frac{1}{n}}$, taking only *a* to the power of *m*

$\Rightarrow \quad \sqrt[n]{a^m} = a^{\frac{m}{n}}$, as before using the property $(r^s)^t = r^{st}$

Summarising the above three results:

If *a* is a real number, and *m* and *n* are two positive integers such that they have no common factors, and that the principal n^{th} root of *a* exists, then

- $a^{\frac{1}{n}} = \sqrt[n]{a}$
- $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$
- $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

Note: If *n* is even, then the domain is $a \geq 0$.

Example:

$$\left(\frac{1}{\sqrt{32}}\right)^{-\frac{2}{5}} = \frac{1^{-\frac{2}{5}}}{\left(\sqrt{32}\right)^{-\frac{2}{5}}}$$, using the property $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

$$= \frac{1}{\left(\sqrt{32}\right)^{-\frac{2}{5}}}$$

$$= \left(\sqrt{32}\right)^{\frac{2}{5}}$$, using property $a^{-n} = \frac{1}{a^n}$ or $a^n = \frac{1}{a^{-n}}$

$$= \left(32^{\frac{1}{2}}\right)^{\frac{2}{5}}$$, from definition $\sqrt[n]{a} = a^{\frac{1}{n}}$

$$= 32^{\frac{1}{2}\times\frac{2}{5}}$$, using property $(a^m)^n = a^{mn}$

$$= 32^{\frac{1}{5}} = \sqrt[5]{32} = 2$$

Rationalising the denominator

- The process of getting rid of the radical in the denominator.

Examples:

1. Rationalise the denominator in $\frac{1}{\sqrt{2}}$

Solution: $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times 1$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$= \frac{\sqrt{2}}{2}$ because $\sqrt{2} \times \sqrt{2} = 2$ in the denominator.

Thus, $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

2. Rationalise the denominator in $\frac{2}{\sqrt[3]{5}}$

Solution: $\frac{2}{\sqrt[3]{5}} = \frac{2}{\sqrt[3]{5}} \times \frac{\sqrt[3]{5}}{\sqrt[3]{5}} \times \frac{\sqrt[3]{5}}{\sqrt[3]{5}}$

$= \frac{2\sqrt[3]{5^2}}{5}$ because $\sqrt[3]{5} \times \sqrt[3]{5} \times \sqrt[3]{5} = 5$ in the denominator.

1.3 Algebraic Expressions

Polynomial

A polynomial is an expression

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0$$

where $a_n, a_{n-1}, a_{n-2}, \cdots, a_2, a_1$ and a_0 are real number coefficients

And n *must be a whole number*

The degree of the polynomial is the highest power of *x* with a non-zero coefficient if the polynomial is not a constant. The degree of a constant function is zero.

Adding and Subtracting Polynomials – We add and subtract **like terms.**

Examples:

1. $x + y + 2x + 3y$
$= (x + 2x) + (y + 3y)$, group like terms
$= 3x + 4y$

2. $(7x^4 - x^2 - 4x + 2) - (3x^4 - 4x^2 + 3x)$
$= 7x^4 - x^2 - 4x + 2 - 3x^4 + 4x^2 - 3x$, make sure we first use the distributive property if necessary
$= (7x^4 - 3x^4) + (-x^2 + 4x^2) + (-4x - 3x) + 2$, group like terms
$= 4x^4 + 3x^2 - 7x + 2$

Multiplying Algebraic Expressions – All the terms in the one of the factors must multiply each and every term in the other factor.

Examples:

1. $(a + b)(c + d)$

$= ac + ad + bc + bd$

2. $(a + b + c)(d + e + f)$

$= ad + ae + af + bd + be + bf + cd + ce + cf$

Examples:

[1] $(2x+1)(x-2) = (2x)(x) + (2x)(-2) + (1)(x) + (1)(-2)$

$$= 2x^2 - 4x + x - 2 = 2x^2 - 3x - 2$$

[2] $(x+2y)(2x-y) = (x)(2x) + (x)(-y) + (2y)(2x) + (2y)(-y)$

$$= 2x^2 - xy + 4xy - 2y^2 = 2x^2 + 3xy - 2y^2$$

[3] $(x-2)(x^2+2x+2) = (x)(x^2) + (x)(2x) + (x)(2) + (-2)(x^2) + (-2)(2x) + (-2)(2)$

$$= x^3 + 2x^2 + 2x - 2x^2 - 4x - 4 = x^3 - 2x - 4$$

Special Product Formulas

Memorise these:

- $(u+v)^2 = u^2 + 2uv + v^2$
- $(u-v)^2 = u^2 - 2uv + v^2$
- $u^2 - v^2 = (u+v)(u-v)$
- $u^3 + v^3 = (u+v)(u^2 - uv + v^2)$
- $u^3 - v^3 = (u-v)(u^2 + uv + v^2)$

Factoring quadratics

Let us say we are given the quadratic $ax^2 + bx + c$. We can rewrite this quadratic as a product of two linear factors

$$ax^2 + bx + c = (mx + r)(nx + s)$$

if the discriminant $b^2 - 4ac$ is a perfect square.

Example:

1. Can $2x^2 - 3x - 2$ be expressed as a product of two linear factors?

Answer: In this example, we have $a = 2$, $b = -3$ and $c = -2$. The discriminant is

$$\begin{aligned} b^2 - 4ac &= (-3)^2 - 4(2)(-2) \\ &= 9 + 16 = 25 \end{aligned}$$

which is a perfect square as $\sqrt{25} = 5$. Therefore, $2x^2 - 3x - 2$ can be expressed as a product of two linear factors.

2. Can $3x^2 - x - 1$ be expressed as a product of two linear factors?

Answer: In this example, we have $a = 3$, $b = -1$ and $c = -1$. The discriminant is

$$\begin{aligned} b^2 - 4ac &= (-1)^2 - 4(3)(-1) \\ &= 1 + 12 = 13 \end{aligned}$$

which is not a perfect square as $\sqrt{13} = 3.60555\cdots$ Therefore, $3x^2 - x - 1$ cannot be expressed as a product of two linear factors.

If a quadratic can be expressed as a product of two linear factors, then we proceed to discuss the following section:

Rewriting $\boldsymbol{ax^2 + bx + c}$ as a product of two linear factors

Step 1: We rewrite $ax^2 + bx + c$ as $ax^2 + px + qx + c$ such that

1. $px + qx = bx$

And 2. $(px)(qx) = (ax^2)(c) = acx^2$; i.e. p and q are factors of ac

So we have $ax^2 + bx + c = ax^2 + px + qx + c$, the middle term rewritten in two terms.

Step 2: Then for every two consecutive terms, we factor out the common factor. If Step 1 was done correctly, we should get two major terms as shown:

$$ax^2 + px + qx + c = nx(mx + r) + s(mx + r)$$

Step 3: Finally, we factor out the common factor from the two major terms.

$$nx(mx + r) + s(mx + r) = (mx + r)(nx + s)$$

In summary, for a quadratic $ax^2 + bx + c$, if its discriminant $b^2 - 4ac$ is a perfect square, we have

$$\begin{aligned} ax^2 + bx + c &= ax^2 + px + qx + c \\ &= nx(mx + r) + s(mx + r) \\ &= (mx + r)(nx + s) \end{aligned}$$

Examples:

1. $2x^2 - 3x - 2$

Step 1: First, we find $(ax^2)(c) = (2x^2)(-2) = -4x^2$.

Next, we find two terms px and qx such that

$px + qx = bx = -3x$

And $(px)(qx) = (ax^2)(c) = -4x^2$

Note: The factors of -4 are ±1, ±2 and ±4.

So let's try $px = x$ and $qx = -4x$

Since $px + qx = x - 4x = -3x$

And $(px)(qx) = (x)(-4x) = -4x^2$

$\therefore$ $px = x$ and $qx = -4x$ work!

Therefore, we rewrite $2x^2 - 3x - 2 = 2x^2 + x - 4x - 2$

Step 2: For every two consecutive terms, we factor out the common factor.

$$2x^2 + x - 4x - 2 = x(2x + 1) - 2(2x + 1)$$

Step 3: We factor out the common factor from the two major terms.

$$x(2x + 1) - 2(2x + 1) = (2x + 1)(x - 2)$$

$$\therefore \quad 2x^2 - 3x - 2 = 2x^2 + x - 4x - 2$$

$$= x(2x + 1) - 2(2x + 1)$$

$$= (2x + 1)(x - 2)$$

Special Factoring Formulas

- $a^2 - b^2 = (a - b)(a + b)$
- $a^2 + 2ab + b^2 = (a + b)^2$
- $a^2 - 2ab + b^2 = (a - b)^2$
- $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
- $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Examples:

[1] $9a^2 - 16 = (3a)^2 - 4^2$

$$= (3a - 4)(3a + 4)$$

[2] $16z^2 - 24z + 9 = (4z)^2 - 2(4z)(3) + 3^2$

$$= (4z - 3)^2$$

Note: Only a very small handful of polynomials follow the pattern of a special factoring formula. Most do not.

1.4 Rational Expressions

Domain – The set of all real numbers that we can possibly assign to a variable.

Two rules for this moment:

[1] If the expression contains even roots :-

The argument in an even root cannot take negative values.

[2] If the expression contains a denominator :-

The denominator cannot be zero.

There are more rules, but we shall come to those later.

Examples:

Find the domain of the expression:

[1] $y = \frac{1}{\sqrt{x-1}}$

Since the argument in an even root must be non-negative and the denominator cannot be zero, therefore we must have

$x - 1 \geq 0$ and $\sqrt{x-1} \neq 0$

$\Rightarrow$ $x \geq 1$ and $x - 1 \neq 0$

$\Rightarrow$ $x \geq 1$ and $x \neq 1$

$\therefore$ $x > 1$ Combining the above two conditions.

Hence, the domain is $x \in (1, \infty)$

[2] $y = \frac{\sqrt{2x}}{x+1}$

The domain must fulfill

$2x \geq 0$ and $x + 1 \neq 0$

$\Rightarrow$ $x \geq 0$ and $x \neq -1$

$\therefore$ $x \geq 0$ Combining the above two conditions.

Hence, the domain is $x \in [0, \infty)$

Simplifying Rational Expressions

Revision: A rational expression has the form $\frac{p}{q}$ where both p and q are polynomials and $q \neq 0$.

In a rational expression, we can cancel common factors, i.e.

$\frac{a\cancel{c}}{b\cancel{c}} = \frac{a}{b}$, provided $b \neq 0$ and $c \neq 0$

Before we can cancel common factors, we should factor completely the numerator and the denominator.

Example:

$$\frac{x^2+4x-12}{3x-6} = \frac{(x+6)\cancel{(x-2)}}{3\cancel{(x-2)}}\,,$$ factoring the numerator and the denominator completely

$$= \frac{x+6}{3}\,,$$ canceling the common factors.

provided $x - 2 \neq 0 \quad \Rightarrow \quad x \neq 2$

Note: It is quite a common mistake among students to want to cancel terms. For example, it is **wrong** to do the following:

$$\frac{(x+2)+\cancel{(x-1)}(x+3)}{2x\cancel{(x-1)}} = \frac{(x+2)+(x+3)}{2x}$$

Multiplying and Dividing Rational Expressions

When multiplying rational expressions, **multiply the numerators and multiply the denominators**.

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

When dividing rational expressions, **invert the divisor and multiply**.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

More examples:

[1] $$\frac{2x^2+x-6}{x^2+4x-5} \times \frac{x^3-3x^2+2x}{4x^2-6x} = \frac{\cancel{(2x-3)}(x+2)}{(x+5)\cancel{(x-1)}} \times \frac{x(x-2)\cancel{(x-1)}}{2x\cancel{(2x-3)}} = \frac{(x+2)(x-2)}{2(x+5)}$$

provided $x + 5 \neq 0$, $x - 1 \neq 0$, $x \neq 0$ and $2x - 3 \neq 0$

$\Rightarrow \quad x \neq -5\,,\ x \neq 1,\ x \neq 0$ and $x \neq \frac{3}{2}$

[2] $\frac{x^3-8}{x^2-4} \div \frac{x^2+2x+4}{x^3+8} = \frac{x^3-8}{x^2-4} \times \frac{x^3+8}{x^2+2x+4} = \frac{(x-2)(x^2+2x+4)}{(x+2)(x-2)} \times \frac{(x+2)(x^2-2x+4)}{x^2+2x+4} = x^2 - 2x + 4$

Provided $x + 2 \neq 0$, $x - 2 \neq 0$ and $x^3 + 8 \neq 0$

$\Rightarrow$ $x \neq -2$, $x \neq 2$ and $x \neq \sqrt[3]{-8}$

Adding or Subtracting rational expressions

The idea is the same as when adding or subtracting fractions, i.e. make the denominators the same so that we can add or subtract the numerators.

If the denominators are not the same, we look for the least common denominator (LCD).

Examples:

[1] $\frac{x}{x-3} - \frac{2}{3x+4} = \frac{x(3x+4)}{(x-3)(3x+4)} - \frac{2(x-3)}{(x-3)(3x+4)}$, make the denominators the same

$= \frac{x(3x+4)-2(x-3)}{(x-3)(3x+4)}$, when the denominators are the same, we can add or subtract the numerators

$= \frac{3x^2+4x-2x+6}{(x-3)(3x+4)} = \frac{3x^2+2x+6}{(x-3)(3x+4)}$

[2] $\frac{3}{x-1} - \frac{2}{x} + \frac{x+3}{x^2-1} = \frac{3}{x-1} - \frac{2}{x} + \frac{x+3}{(x-1)(x+1)}$, remember to completely factor the denominators before we begin making the denominators the same

$$= \frac{3(x)(x+1)}{(x-1)(x)(x+1)} - \frac{2(x-1)(x+1)}{(x-1)(x)(x+1)} + \frac{(x+3)(x)}{(x-1)(x)(x+1)}$$

$$= \frac{3x(x+1) - 2(x-1)(x+1) + (x+3)x}{(x-1)(x)(x+1)} = \frac{3x^2 + 3x - 2(x^2-1) + x^2 + 3x}{(x-1)(x)(x+1)}$$

$$= \frac{3x^2 + 3x - 2x^2 + 2 + x^2 + 3x}{(x-1)(x)(x+1)} = \frac{2x^2 + 6x + 2}{(x-1)(x)(x+1)} = \frac{2(x^2 + 3x + 1)}{(x-1)(x)(x+1)}$$

Compound fractions

These are rational expressions where the numerator or the denominator or both are rational expressions themselves.

One method is to rewrite the numerator and the denominator into single fractions respectively before simplifying the rational expression.

Example:

[1] $\dfrac{\frac{2}{x}-3}{1-\frac{1}{x-1}} = \dfrac{\frac{2}{x}-\frac{3x}{x}}{\frac{x-1}{x-1}-\frac{1}{x-1}} = \dfrac{\frac{2-3x}{x}}{\frac{x-1-1}{x-1}} = \dfrac{\frac{2-3x}{x}}{\frac{x-2}{x-1}}$, simplifying the numerator and the denominator respectively

$= \dfrac{2-3x}{x} \div \dfrac{x-2}{x-1}$, an expression over another expression is basically a division

$= \dfrac{2-3x}{x} \times \dfrac{x-1}{x-2} = \dfrac{(2-3x)(x-1)}{x(x-2)}$

OR We find the least common denominator of all the fractions in the numerator and the denominator. Next, we multiply and divide the compound fraction with this result.

Example: (Same problem as above.)

[2] $\dfrac{\frac{2}{x}-3}{1-\frac{1}{x-1}} = \dfrac{\left(\frac{2}{x}-3\right)(x)(x-1)}{\left(1-\frac{1}{x-1}\right)(x)(x-1)}$, the least common denominator in the numerator and the denominator is $(x)(x-1)$

$= \dfrac{\frac{2}{x}(x)(x-1)-3(x)(x-1)}{(x)(x-1)-\frac{1}{x-1}(x)(x-1)}$, distribute in the numerator and the denominator

$= \dfrac{2(x-1)-3x(x-1)}{x(x-1)-x} = \dfrac{(x-1)(2-3x)}{x\left((x-1)-1\right)} = \dfrac{(x-1)(2-3x)}{x(x-2)}$

Simplifying an expression with negative and/or rational exponents

Keys:

1. When we factor out a common factor from terms in an expression, we always take out the factor with the smallest exponent.

2. What remains is the difference in the exponents. Suppose $n < m$,

$$a^m + a^n = a^n(a^{m-n} + 1)$$

Factor out the common factor with the smallest exponent.

What remains is the difference in the exponents.

Examples:

[1] $3x^{-\frac{5}{2}}+2x^{-\frac{3}{2}}=x^{-\frac{5}{2}}\left(3+2x^{-\frac{3}{2}-\left(-\frac{5}{2}\right)}\right)$, factor out the factor with the smaller exponent. Here, $-\frac{5}{2}<-\frac{3}{2}$. What remains is the difference in the exponents.

$$=x^{-\frac{5}{2}}\left(3+2x^{-\frac{3}{2}+\frac{5}{2}}\right)=x^{-\frac{5}{2}}\left(3+2x^{\frac{2}{2}}\right)=x^{-\frac{5}{2}}(3+2x)$$

[2] $\frac{(4-x^2)^{\frac{1}{2}}+x^2(4-x^2)^{-\frac{1}{2}}}{4-x^2}=(4-x^2)^{\frac{1}{2}-1}+x^2(4-x^2)^{-\frac{1}{2}-1}$

$$=(4-x^2)^{-\frac{1}{2}}+x^2(4-x^2)^{-\frac{3}{2}}=(4-x^2)^{-\frac{3}{2}}\left((4-x^2)^{-\frac{1}{2}-\left(-\frac{3}{2}\right)}+x^2\right)$$

$$=(4-x^2)^{-\frac{3}{2}}((4-x^2)^1+x^2)=(4-x^2)^{-\frac{3}{2}}(4)=\frac{4}{(4-x^2)^{\frac{3}{2}}}=\frac{4}{\sqrt{(4-x^2)^3}}$$

[3] $4x^3(2x-1)^{\frac{3}{2}}-2x(2x-1)^{-\frac{1}{2}}=2x(2x-1)^{-\frac{1}{2}}\left(2x^{3-1}(2x-1)^{\frac{3}{2}-\left(-\frac{1}{2}\right)}-1\right)$

$$=2x(2x-1)^{-\frac{1}{2}}(2x^2(2x-1)^2-1)=\frac{2x(2x^2(2x-1)^2-1)}{(2x-1)^{\frac{1}{2}}}$$

Rationalising the Denominator or the Numerator

What is a conjugate?

$a\pm b$ and $a\mp b$ are conjugates of each other.

Examples: The conjugate of $x+y$ is $x-y$ or $-x+y$,
The conjugate of $r-t$ is $r+t$ or $-r-t$.

To obtain the conjugate of a two-term expression, simply change the sign of one of the terms.

Rule of thumb:

- If we wish to rationalise the numerator, we multiply and divide the rational with the conjugate of the numerator.
- If we wish to rationalise the denominator, we multiply and divide the rational with the conjugate of the denominator.

Examples:

Rationalise the denominator:

[1] $\frac{2(x-y)}{\sqrt{x}-\sqrt{y}} = \frac{2(x-y)}{\sqrt{x}-\sqrt{y}} \times \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}}$; One of the conjugates of $\sqrt{x} - \sqrt{y}$ is $\sqrt{x} + \sqrt{y}$.

$$= \frac{2(x-y)(\sqrt{x}+\sqrt{y})}{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})} = \frac{2(x-y)(\sqrt{x}+\sqrt{y})}{x-y} = 2(\sqrt{x}+\sqrt{y})$$

Rationalise the numerator:

[2] $\frac{\sqrt{x+h+1}-\sqrt{x+1}}{h} = \frac{\sqrt{x+h+1}-\sqrt{x+1}}{h} \times \frac{\sqrt{x+h+1}+\sqrt{x+1}}{\sqrt{x+h+1}+\sqrt{x+1}}$, a conjugate of $\sqrt{x+h+1} - \sqrt{x+1}$ is $\sqrt{x+h+1} + \sqrt{x+1}$

$$= \frac{\left(\sqrt{x+h+1}\right)^2 - \left(\sqrt{x+1}\right)^2}{h\left(\sqrt{x+h+1}+\sqrt{x+1}\right)} = \frac{(x+h+1)-(x+1)}{h\left(\sqrt{x+h+1}+\sqrt{x+1}\right)} = \frac{x+h+1-x-1}{h\left(\sqrt{x+h+1}+\sqrt{x+1}\right)}$$

$$= \frac{h}{h\left(\sqrt{x+h+1}+\sqrt{x+1}\right)} = \frac{1}{\sqrt{x+h+1}+\sqrt{x+1}}$$

1.5 Equations

General Rule of Thumb when treating both sides of an equation: Be fair to both sides.

1. If we add a value to one side of an equation, we have to add the same value to the other side.

Example: $a = b$
$\Rightarrow$ $a + c = b + c$

2. If we subtract a value from one side of an equation, we have to subtract the same value from the other side.

Example: $a = b$
$\Rightarrow$ $a - d = b - d$

3. If we multiply a value to one side of an equation, we have to multiply the same value to the other side.

Example: $a = b$
$\Rightarrow$ $ac = bc$

4. If we divide one side of an equation by a non-zero value, we have to divide the other side by the same value.

Example: $a = b$
$\Rightarrow$ $\frac{a}{c} = \frac{b}{c}$, provided $c \neq 0$

General Rule of Thumb when moving a term from one side of the equation to the other.

When we move a term, we change its sign, i.e. positive becomes negative and negative becomes positive.

1. $a + b = c$
$\Rightarrow$ $a = c - b$

Proof: $a + b = c$
$\Rightarrow$ $a + b - b = c - b$
$\Rightarrow$ $a = c - b$

2. $a - b = c$
$\Rightarrow$ $a = c + b$

Proof: $a - b = c$
$\Rightarrow$ $a - b + b = c + b$
$\Rightarrow$ $a = c + b$

General Rule of Thumb when moving a factor from one side of the equation to the other.

When we move a factor from one side of the equation to the other, if it was originally in the numerator, we move it to the denominator at the other side. And if it was originally in the denominator, we move it to the numerator at the other side.

1. $\frac{a}{b} = \frac{c}{d}$

$\Rightarrow \quad \frac{1}{b} = \frac{c}{ad}$

Proof: $\frac{a}{b} = \frac{c}{d}$

$\Rightarrow \quad \frac{a}{b} \times \frac{1}{a} = \frac{c}{d} \times \frac{1}{a}$

$\Rightarrow \quad \frac{1}{b} = \frac{c}{ad}$

2. $\frac{a}{b} = \frac{c}{d}$

$\Rightarrow \quad a = \frac{bc}{d}$

Proof: $\frac{a}{b} = \frac{c}{d}$

$\Rightarrow \quad \frac{a}{b} \times b = \frac{c}{d} \times b$

$\Rightarrow \quad a = \frac{bc}{d}$

Example:

Solve the equation for the indicated variable:

[1] $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$; for R_1

We wish to isolate the variable R_1, i.e. rewrite the equation as R_1 in terms of the other variables.

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

$$\Rightarrow \quad \frac{1}{R_1} = \frac{1}{R} - \frac{1}{R_2} = \frac{R_2}{R_2 R} - \frac{R}{R_2 R}$$

$$\Rightarrow \quad \frac{1}{R_1} = \frac{R_2 - R}{R_2 R}$$

$\Rightarrow \quad R_1 = \frac{R_2 R}{R_2 - R}$; Taking the reciprocals of both sides.

Solving Linear Equations

In general, to solve a linear equation, these are the steps:

$$ax + b = 0$$

$\Rightarrow$ $\quad ax = -b$

$\Rightarrow$ $\quad x = -\frac{b}{a}$

Examples: In the following exercises, solve for *x*.

[1] $\quad 4x + 7 = 9x - 3$

$\Rightarrow$ $\quad 4x - 9x = -3 - 7$ $\quad$ Move all terms with *x* to one side of the equation, and all terms without *x* to the other side.

$\Rightarrow$ $\quad -5x = -10$

$\therefore$ $\quad x = \frac{-10}{-5} = 2$

[2] $\quad 1 - \big(2 - (3 - x)\big) = 4x - (6 + x)$

$\Rightarrow$ $\quad 1 - (2 - 3 + x) = 4x - 6 - x$

$\Rightarrow$ $\quad 1 - (-1 + x) = 3x - 6$

$\Rightarrow$ $\quad 1 + 1 - x = 3x - 6$

$\Rightarrow$ $\quad 2 - x = 3x - 6$

$\Rightarrow$ $\quad -x - 3x = -6 - 2$

$\Rightarrow$ $\quad -4x = -8$

$\therefore$ $\quad x = \frac{-8}{-4} = 2$

Solving Quadratic Equations

Completing the Square and Solving a Quadratic Equation

[1] $\quad (3x + 2)^2 = 10$

$\Rightarrow$ $\quad 3x + 2 = \pm\sqrt{10}$

$\Rightarrow \qquad 3x + 2 = \sqrt{10}$ or $3x + 2 = -\sqrt{10}$

$\Rightarrow \qquad 3x = -2 + \sqrt{10}$ or $3x = -2 - \sqrt{10}$

$\Rightarrow \qquad x = \frac{-2+\sqrt{10}}{3}$ or $\frac{-2-\sqrt{10}}{3}$

[2] $\qquad 3x^2 - 6x - 1 = 0$

$\Rightarrow \qquad 3(x^2 - 2x) - 1 = 0$; Take the coefficient of *x* within the brackets, halve it and square the result, i.e. $\left(\frac{-2}{2}\right)^2 = (-1)^2 = 1$

$\Rightarrow \qquad 3(x^2 - 2x + 1 - 1) - 1 = 0$; Add and subtract the above result within the brackets.

$\Rightarrow \qquad 3(x^2 - 2x + 1) + (3)(-1) - 1 = 0$

$\Rightarrow \qquad 3(x - 1)^2 - 3 - 1 = 0$

$\Rightarrow \qquad 3(x - 1)^2 - 4 = 0$

$\Rightarrow \qquad 3(x - 1)^2 = 4$

$\Rightarrow \qquad (x - 1)^2 = \frac{4}{3}$

$\Rightarrow \qquad x - 1 = \pm\sqrt{\frac{4}{3}} = \pm\frac{2}{\sqrt{3}} = \pm\frac{2\sqrt{3}}{3}$

$\Rightarrow \qquad x - 1 = \frac{2\sqrt{3}}{3}$ or $x - 1 = -\frac{2\sqrt{3}}{3}$

$\therefore \qquad x = 1 + \frac{2\sqrt{3}}{3}$ or $x = 1 - \frac{2\sqrt{3}}{3}$

The Quadratic Formula

Recall that: A quadratic equation has the form

$$ax^2 + bx + c = 0$$

And has solutions given by the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The number $b^2 - 4ac$ is called the discriminant and there are three possibilities based on the value of the discriminant:

- If $b^2 - 4ac > 0$, then the quadratic has two real and distinct solutions.
- If $b^2 - 4ac = 0$, then the quadratic equation has one real solution.
- If $b^2 - 4ac < 0$, then the quadratic has no real solutions.

Examples:

[1] $3x^2 - 6x - 1 = 0$

$\Rightarrow$ $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ where $a = 3$, $b = -6$ and $c = -1$

$$= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(-1)}}{2(3)} = \frac{6 \pm \sqrt{36+12}}{6} = \frac{6 \pm \sqrt{48}}{6} = \frac{6 \pm \sqrt{(16)(3)}}{6} = \frac{6 \pm \sqrt{16}\sqrt{3}}{6}$$

$$= \frac{6 \pm 4\sqrt{3}}{6} = \frac{6}{6} \pm \frac{4\sqrt{3}}{6} = 1 \pm \frac{2\sqrt{3}}{3}$$

$\Rightarrow$ $x = 1 + \frac{2\sqrt{3}}{3}$ or $x = 1 - \frac{2\sqrt{3}}{3}$

Zero product property

$AB = 0$ if and only if $A = 0$ or $B = 0$

Corollary: If the quadratic expression can be factorised as follows:

$$ax^2 + bx + c = (mx + r)(nx + s)$$

Then, we have

$ax^2 + bx + c = 0$

$\Rightarrow$ $(mx + r)(nx + s) = 0$

$\Rightarrow$ $mx + r = 0$ or $nx + s = 0$

$\Rightarrow$ $mx = -r$ or $nx = -s$

$\Rightarrow$ $x = -\frac{r}{m}$ or $x = -\frac{s}{n}$

Examples:

[1] $6x(x-1) = 21 - x$

$\Rightarrow \quad 6x^2 - 6x = 21 - x$

$\Rightarrow \quad 6x^2 - 6x - 21 + x = 0$

$\Rightarrow \quad 6x^2 - 5x - 21 = 0$

The discriminant is $b^2 - 4ac = (-5)^2 - 4(6)(-21) = 25 + 504 = 529 = 23^2$, a perfect square.

$\therefore \quad 6x^2 - 5x - 21$ can be factorised into two linear factors with integer coefficients.

Step #1: We want to find px and qx such that

$px + qx = bx = -5x$

And $(px)(qx) = (ax^2)(c) = (6x^2)(-21) = -126x^2$

The factors of -126 are ± 1, ± 2, ± 3, ± 6, ± 7, ± 9, ± 14, ± 18, ± 21, ± 42, ± 63 and ± 126.
These are the possible values of *p* and *q*.
Also, we take note that pq is a negative number, meaning *p* and *q* are of different signs.

Let's try $px = -14x$ and $qx = 9x$

Check: $px + qx = -14x + 9x = -5x$

And $(px)(qx) = (-14x)(9x) = -126x^2$

Both conditions are fulfilled, so $px = -14x$ and $qx = 9x$ work!

So, $6x^2 - 5x - 21 = 6x^2 - 14x + 9x - 21$

$= 2x(3x - 7) + 3(3x - 7)$

$= (3x - 7)(2x + 3)$

$\therefore \quad 6x^2 - 5x - 21 = 0$

$\Rightarrow \quad (3x - 7)(2x + 3) = 0$

$\Rightarrow \quad 3x - 7 = 0$ or $2x + 3 = 0$

$\Rightarrow \quad 3x = 7$ or $2x = -3$

$\Rightarrow \quad x = \frac{7}{3}$ or $x = -\frac{3}{2}$

[2] $\frac{1}{x} - \frac{1}{x-4} = 1$

$\Rightarrow \quad \frac{x-4}{x(x-4)} - \frac{x}{x(x-4)} = 1$

$\Rightarrow \quad \frac{x-4-x}{x(x-4)} = 1$

$\Rightarrow \quad \frac{-4}{x(x-4)} = 1$

$\Rightarrow \quad -4 = x(x-4) = x^2 - 4x$

$\Rightarrow \qquad x^2 - 4x + 4 = 0$

$\Rightarrow \qquad (x-2)^2 = 0$

$\Rightarrow \qquad x - 2 = 0$

$\therefore \qquad x = 2$

[3] $\qquad z^2 - \frac{3}{2}z + \frac{9}{16} = 0$

$\Rightarrow \qquad 16\left(z^2 - \frac{3}{2}z + \frac{9}{16}\right) = 16(0)$

$\Rightarrow \qquad 16z^2 - 24z + 9 = 0$

$\Rightarrow \qquad (4z-3)^2 = 0$

$\Rightarrow \qquad 4z - 3 = 0$

$\Rightarrow \qquad z = \frac{3}{4}$

[4] $\qquad \sqrt{2x+1} + 1 = x$

$\Rightarrow \qquad \sqrt{2x+1} = x - 1$

$\Rightarrow \qquad \left(\sqrt{2x+1}\right)^2 = (x-1)^2$; Square both sides.

$\Rightarrow \qquad 2x + 1 = x^2 - 2x + 1$

$\Rightarrow \qquad 2x + 1 - x^2 + 2x - 1 = 0$

$\Rightarrow \qquad -x^2 + 4x = 0$

$\Rightarrow \qquad x(-x+4) = 0$

$\Rightarrow \qquad x = 0$ or $-x + 4 = 0$

$\Rightarrow \qquad x = 0$ or $x = 4$ These are the possible solutions.

Important: When solving an equation with an even root, it is important to plug in our results in the original equation to see if we have to reject any of the possible solutions.

So let us plug in $x = 0$ as well as $x = 4$ into the equation $\sqrt{2x+1} + 1 = x$ and see which one works.

When $x = 0$, we have $\sqrt{2(0)+1} + 1 = 0$

$\Rightarrow \qquad \sqrt{1} + 1 = 0$

$\Rightarrow \qquad 2 = 0$ which is clearly incorrect!!!

Therefore, $x = 0$ cannot be a solution.

When $x = 4$, we have $\sqrt{2(4) + 1} + 1 = 4$

$\Rightarrow \qquad \sqrt{8 + 1} + 1 = 4$

$\Rightarrow \qquad \sqrt{9} + 1 = 4$

$\Rightarrow \qquad 3 + 1 = 4$ which is clearly correct.

Therefore, $x = 4$ is a solution.

$\therefore \qquad \sqrt{2x + 1} + 1 = x$ has one solution, i.e. $x = 4$.

[5] $\qquad |3x + 5| = 1$

$\Rightarrow \qquad 3x + 5 = -1$ or $3x + 5 = 1$

$\Rightarrow \qquad 3x = -1 - 5$ or $3x = 1 - 5$

$\Rightarrow \qquad 3x = -6$ or $3x = -4$

$\Rightarrow \qquad x = -\frac{6}{3}$ or $x = -\frac{4}{3}$

$\Rightarrow \qquad x = -2$ or $x = -\frac{4}{3}$

OR $\qquad |3x + 5| = 1$

$\Rightarrow \qquad (3x + 5)^2 = 1^2$

$\Rightarrow \qquad 9x^2 + 30x + 25 = 1$

$\Rightarrow \qquad 9x^2 + 30x + 25 - 1 = 0$

$\Rightarrow \qquad 9x^2 + 30x + 24 = 0$

$\Rightarrow \qquad 3(3x^2 + 10x + 8) = 0$

$\Rightarrow \qquad 3(3x + 4)(x + 2) = 0$

$\Rightarrow \qquad 3x + 4 = 0$ or $x + 2 = 0$

$\Rightarrow \qquad 3x = -4$ or $x = -2$

$\Rightarrow \qquad x = -\frac{4}{3}$ or $x = -2$

1.6 Modelling with Equations

When we are given a situation to solve, we need to do is to come up with an equation that models the problem at hand. To find the solution to the problem, we solve this equation. More often than not, drawing a diagram of the situation helps a lot too.

Examples:

[1] **Renting a Truck:** A rental company charges \$65 a day and 20 cents a mile for renting a truck. Stavros rented a truck for 3 days, and his bill came to \$275. How many miles did he drive?

Solution: Let x be the number of miles Stavros drove.

We are told that Stavros rented the truck for 3 days at \$65 per day, so the basic rental for the 3 days would be

$$Basic\ rental = No.of\ days\ \times Rental\ per\ day$$

$$= 3 \times \$65 = \$195$$

We are also told that the mileage charges were 20 cents per mile, hence the total mileage charges that Stavros incurred would be

$$Mileage\ charges = Miles\ driven \times Charge\ per\ mile$$

$$= x \times 20\ cents$$

$$= x \times \$0.20 = \$0.2x$$

Where *x* is the number of miles that Stavros drove with the truck.

We are also told his total bill came up to \$275. This total bill would be the sum of the basic rental and the mileage charges, i.e.

$$Total\ rental = Basic\ rental + Mileage\ charges$$

$$\Rightarrow \quad \$275 = \$195 + \$0.2x$$

$$\Rightarrow \quad 0.2x = 275 - 195$$

$$= 80$$

$$\Rightarrow \quad x = \frac{80}{0.2} = \frac{800}{2} = 400\ miles$$

[2] **Investments** Amber invested \$12,000, a portion earning a simple interest rate of $4\frac{1}{2}\%$ per year and the rest earning a rate of 4% per year. After 1 year the total interest earned on these investments was \$525. How much money did she invest at each rate?

Solution: Let x be the amount earning $4\frac{1}{2}\%$ per year
And Let y be the amount earning 4% per year

We know that the total amount invested is $x + y = \$12000$ ---(1)

And we are told that the total interest earned was \$525, i.e.

$$Interest\ earned\ on\ the\ x\ portion + Interest\ earned\ on\ the\ y\ portion = \$525$$

$\Rightarrow \quad x \times 4\frac{1}{2}\% + y \times 4\% = \$525$

$\Rightarrow \quad x \times 0.045 + y \times 0.04 = \$525$

$\Rightarrow \quad 0.045x + 0.04y = \$525$ ---(2)

We have two equations in two unknowns that we can solve:

From (1) $\quad x + y = \$12000$

$\Rightarrow \quad y = \$12000 - x$ ---(3)

Substitute (3) into (2),

$0.045x + 0.04y = \$525$

$\Rightarrow \quad 0.045x + 0.04(\$12000 - x) = \$525$

$\Rightarrow \quad 0.045x + \$480 - 0.04x = \$525$

$\Rightarrow \quad 0.005x = \$525 - \$480$

$= \$45$

$\Rightarrow \quad x = \frac{\$45}{0.005} = \frac{\$45000}{5} = \$9000$ ---(4)

Substitute (4) into (3) to obtain *y*,

$y = \$12000 - x$

$= \$12000 - \$9000 = \$3000$

$\therefore$ \$9000 was earning $4\frac{1}{2}\%$ interest and \$3000 was earning 4% interest.

[3] **A Riddle** A movie star, unwilling to give his age, posed the following riddle to a gossip columnist: "Seven years ago, I was eleven times as old as my daughter. Now I am four times as old as she is." How old is the movie star?

Solution: Let *m* be the age of the movie star
And Let *d* be the age of his daughter

We are told the movie star is four times as old as his daughter, i.e.

$$m = 4d \qquad \text{---(1)}$$

Seven years ago, the movie star was 11 times as old as his daughter. Seven years ago, the ages of the movie star and his daughter would be $m - 7$ and $d - 7$ respectively. So,

$$m - 7 = 11(d - 7)$$

$$= 11d - 77$$

$$\Rightarrow \quad m = 11d - 77 + 7$$

$$\Rightarrow \quad m = 11d - 70 \qquad \text{---(2)}$$

Substitute (1) into (2)

$$m = 11d - 70$$

$$\Rightarrow \quad 4d = 11d - 70$$

$$\Rightarrow \quad 4d - 11d = -70$$

$$\Rightarrow \quad -7d = -70$$

$$\Rightarrow \quad d = \frac{-70}{-7} = 10 \qquad \text{---(3)}$$

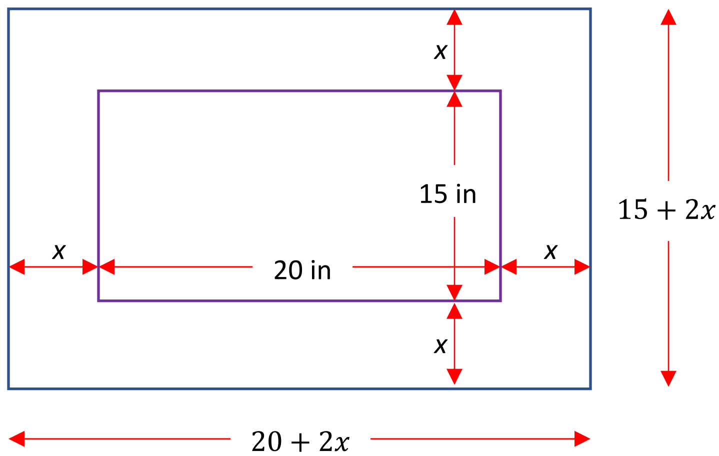
Substitute (3) into (1) to solve for *m*,

$$m = 4d = 4 \times 10 = 40$$

∴ The movie star is currently 40 years old and his daughter is 10 years old.

[4] **Framing a Painting** Jack paints with watercolours on a sheet of paper 20 in. wide by 15 in. high. He then places this sheet on a mat so that a uniformly wide strip of the mat shows all around the picture. The perimeter of the mat is 102 in. How wide is the strip of the mat showing around the picture?

Solution: Let *x* be the width of the strip.



We are told the perimeter of the mat is 102 in. We can thus write that

$$2(20 + 2x) + 2(15 + 2x) = 102$$

$\Rightarrow$ $2(20 + 2x + 15 + 2x) = 102$

$\Rightarrow$ $20 + 2x + 15 + 2x = \frac{102}{2} = 51$

$\Rightarrow$ $4x + 35 = 51$

$\Rightarrow$ $4x = 51 - 35 = 16$

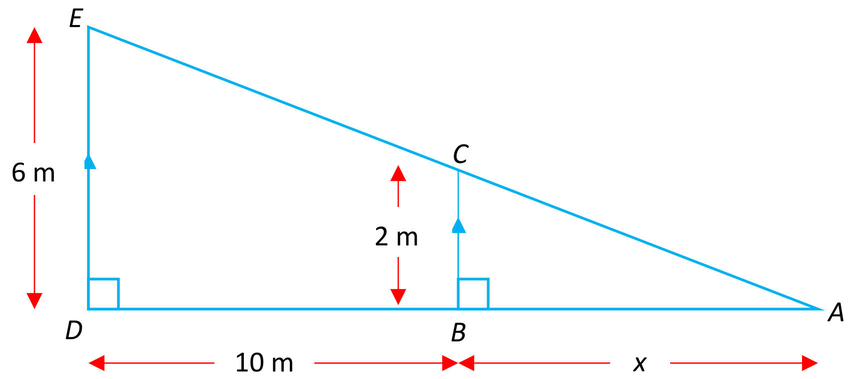
$\Rightarrow$ $x = \frac{16}{4} = 4\ in.$

$\therefore$ The width of the strip is 4 in.

[5] **Length of a Shadow** A man is walking away from a lamppost with a light source 6 m above the ground. The man is 2 m tall. How long is the man's shadow when he is 10 m from the lamppost? [*Hint:* Use similar triangles.]

Solution: Redrawing the diagram:

Let *x* be the length of the shadow:



Revisiting similar triangles, if BC and DE are parallel, we have

$$\frac{AB}{BC} = \frac{AD}{DE}$$

$\Rightarrow$ $\frac{x}{2} = \frac{10+x}{6}$

$\Rightarrow$ $6x = 2(10 + x) = 20 + 2x$

$\Rightarrow$ $6x - 2x = 20$

$\Rightarrow$ $4x = 20$

$\Rightarrow$ $x = \frac{20}{4} = 5\ m$

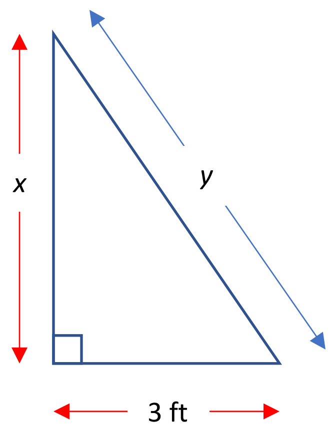
$\therefore$ His shadow is 5 m long.

[6] **An ancient Chinese problem** This problem is taken from a Chinese mathematics textbook called 九章算术, or *The Nine Chapters on the Mathematical Art*, which was compiled from the 10^{th} century BCE to the 2^{nd} century CE.

A 10-ft-long stem of bamboo is broken in such a way that its tip touches the ground 3 ft from the base of the stem, as shown in the figure. What is the height of the break?

Solution: Let us redraw the diagram:

Let x be the height of the triangle,
And Let y be the hypotenuse.



We are told that the bamboo is 10 ft long, i.e.

$$x + y = 10 \qquad \text{---(1)}$$

Using Pythagoras' Theorem, we have

$$x^2 + 3^2 = y^2 \qquad \text{---(2)}$$

From (1), we can rewrite as y in terms of x,

$$x + y = 10$$

$$\Rightarrow \quad y = 10 - x \qquad \text{---(3)}$$

Substitute (3) into (2),

$$x^2 + 3^2 = y^2$$

$$\Rightarrow \quad x^2 + 9 = (10 - x)^2$$

$$= 100 - 20x + x^2$$

$$\Rightarrow \quad 9 = 100 - 20x$$

$\Rightarrow \qquad 20x = 100 - 9 = 91$
$\Rightarrow \qquad x = \frac{91}{20} = 4\frac{11}{20}\ ft$

$\therefore \qquad$ The height of the break is $4\frac{11}{20}\ ft$.

Note: Ancient Chinese mathematicians had discovered the Pythagoras Theorem independently from the Greek mathematicians. They called this theorem the 勾股 (Gou Gu) Theorem. Numerous articles have been written about this, one of which is
http://5010.mathed.usu.edu/Fall2014/BProbst/TheChinese.html .

1.7 Inequalities

Properties of Inequalities

Let *a*, *b*, *c* and *d* be real numbers. We have the following properties:

1. **Transitive Property**

   If $a < b$ and $b < c$, then $a < c$.

   If $a \leq b$ and $b \leq c$, then $a \leq c$.

2. **Addition of Inequalities**

   If $a < b$ and $c < d$, then $a + c < b + d$

   If $a \leq b$ and $c \leq d$, then $a + c \leq b + d$

3. **Addition of a Constant**

   If $a < b$, then $a + c < b + c$

   If $a \leq b$, then $a + c \leq b + c$

4. **Multiplication by a Constant**

   If $c > 0$ and $a < b$, then $ac < bc$

   If $c > 0$ and $a \leq b$, then $ac \leq bc$

   If $c < 0$ and $a < b$, then $ac > bc$

   If $c < 0$ and $a \leq b$, then $ac \geq bc$

Important: When we multiply or divide by a negative number, we must reverse the inequality sign.

Solving inequalities

Examples:

1. $5x - 7 < 3x + 9$

$\Rightarrow$ $5x - 3x < 9 + 7$, same rules of moving terms apply

$\Rightarrow$ $2x < 16$

$\Rightarrow \quad x < \frac{16}{2}$, When dividing by a positive number, we do not reverse the inequality sign.

$\Rightarrow \quad x < 8$ or $x \in (-\infty, 8)$

OR

$5x - 7 < 3x + 9$

$\Rightarrow \quad 3x + 9 > 5x - 7$

$\Rightarrow \quad 3x - 5x > -7 - 9$

$\Rightarrow \quad -2x > -16$

$\Rightarrow \quad x < \frac{-16}{-2}$, When dividing by a negative number, we have to reverse the inequality sign.

$\Rightarrow \quad x < 8$

2. $1 - \frac{3}{2}x \geq x - 4$

$\Rightarrow \quad 2 - 3x \geq 2x - 8$, Multiply every term by 2 to avoid working with fractions.

$\Rightarrow \quad -3x - 2x \geq -8 - 2$

$\Rightarrow \quad -5x \geq -10$

$\Rightarrow \quad x \leq \frac{-10}{-5}$, Remember, when dividing by a negative number, we reverse the inequality sign.

$\Rightarrow \quad x \leq 2$ or $x \in (-\infty, 2]$

3. $-3 \leq 6x - 1 < 3$

$\Rightarrow \quad -3 + 1 \leq 6x < 3 + 1$

$\Rightarrow \quad -2 \leq 6x < 4$

$\Rightarrow \quad \frac{-2}{6} \leq x < \frac{4}{6}$

$\Rightarrow \quad -\frac{1}{3} \leq x < \frac{2}{3}$ or $x \in \left[-\frac{1}{3}, \frac{2}{3}\right)$

Quadratic Inequality

We know that all quadratics yield graphs that are parabolic. So when we solve inequalities involving quadratics, we can look at the quadratic graph, i.e. the parabola, to solve quadratic inequalities.

Examples: Find intervals that satisfy the following inequalities:

1. Solve $4x^2 - 5x - 6 > 0$

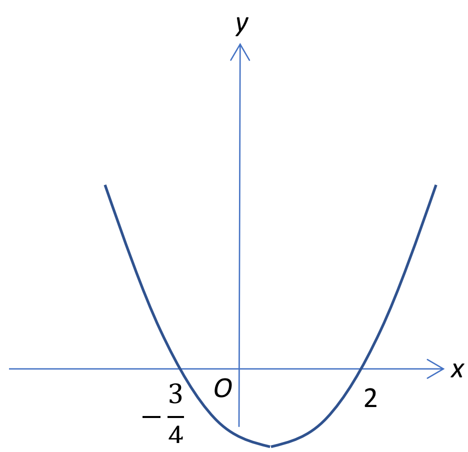
Step #1: Assume that $f(x) = 4x^2 - 5x - 6 = 0$

$\Rightarrow \quad (4x + 3)(x - 2) = 0$

$\Rightarrow \quad 4x + 3 = 0$ or $x - 2 = 0$

$\Rightarrow \quad x = -\frac{3}{4}$ or $x = 2$

Step #2: The graph of $f(x) = 4x^2 - 5x - 6$ is a concave upward parabola that intersects the x-axis at $x = -\frac{3}{4}$ and $x = 2$.



We want the intervals when $f(x) = 4x^2 - 5x - 6 > 0$, i.e. we want intervals for the portions of the graph when $f(x) > 0$ or when $y > 0$. In other words, we want the intervals for the portions of the graph that are above the *x*-axis. From the graph, we can see that the portions of the graph that are above the *x*-axis occur on the intervals $x < -\frac{3}{4}$ and $x > 2$.

∴ The solution to $4x^2 - 5x - 6 > 0$ is $x \in \left(-\infty, -\frac{3}{4}\right) \cup (2, \infty)$.

2. Solve $2x - x^2 \geq 0$

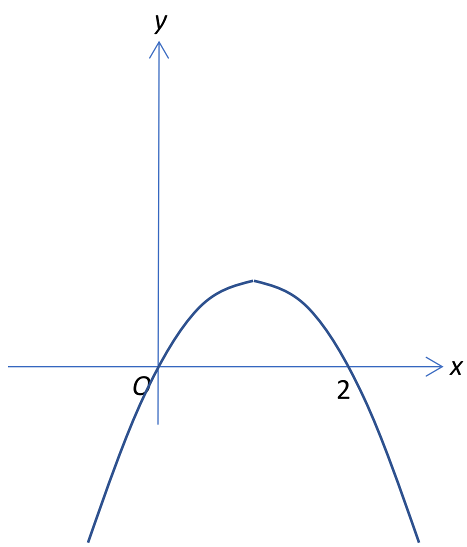
Step #1: Assume $f(x) = 2x - x^2 = 0$

$\Rightarrow \quad x(2 - x) = 0$

$\Rightarrow \quad x = 0$ or $2 - x = 0$

$\Rightarrow \quad x = 0$ or $x = 2$

Step #2: The graph of $f(x) = 2x - x^2$ is a concave downward parabola that intersects the *x*-axis at $x = 0$ and $x = 2$.



We want the intervals when $f(x) = 2x - x^2 \geq 0$, i.e. we want intervals for the portions of the graph when $f(x) \geq 0$ or when $y \geq 0$. In other words, we want the intervals for the portions of the graph that are above and on the *x*-axis. From the graph, we can see that one portion of the graph that is above and on the *x*-axis occurs on the interval $0 \leq x < 2$.

∴ The solution to $2x - x^2 \geq 0$ is $x \in [0,2]$.

Polynomial Inequalities

In this section, we discuss a method of using a Sign Table to solve polynomial inequalities.

Example: Solve $2x^3 - 3x^2 - 32x + 48 > 0$

Let $f(x) = 2x^3 - 3x^2 - 32x + 48$

Step #1: Assume $2x^3 - 3x^2 - 32x + 48 = 0$

$\Rightarrow \quad (2x - 3)(x - 4)(x + 4) = 0$

$\Rightarrow \qquad 2x - 3 = 0$ or $x - 4 = 0$ or $x + 4 = 0$

$\Rightarrow \qquad x = \frac{3}{2}$ or $x = 4$ or $x = -4$

Step #2: We now draw a sign table on the number line using the above values of *x* as divisions, i.e. we divide the number line into intervals $x < -4$, $-4 < x < \frac{3}{2}$, $\frac{3}{2} < x < 4$ and $x > 4$. Then, on each interval, we choose a value of *x* as a test value that we will assign into the polynomial $f(x)$. The sign of the polynomial on each interval indicates whether that portion of the graph of the polynomial is above or below the *x*-axis.

Intervals	$(-\infty, -4)$	$\left(-4, \frac{3}{2}\right)$	$\left(\frac{3}{2}, 4\right)$	$(4, \infty)$
Test Value	$x = -5$	$x = 0$	$x = 2$	$x = 5$
Sign of $2x - 3$	$-ve$	$-ve$	$+ve$	$+ve$
Sign of $x - 4$	$-ve$	$-ve$	$-ve$	$+ve$
Sign of $x + 4$	$-ve$	$+ve$	$+ve$	$+ve$
Sign of $f(x) = (2x - 3)(x - 4)(x + 4)$	$-ve$	$+ve$	$-ve$	$+ve$

$\therefore$ From the last line in the above table, we can conclude that the solution to

$2x^3 - 3x^2 - 32x + 48 > 0$ is $x \in \left(-4, \frac{3}{2}\right) \cup (4, \infty)$.

Here is an application: Find the domain of the expression $\sqrt{3x^2(x-1)}$.

We know that the argument in an even root must be non-negative, i.e. the domain is the solution to the inequality $3x^2(x-1) \geq 0$.

Let $f(x) = 3x^2(x-1)$.

Step #1: Assume $3x^2(x-1) = 0$

$\Rightarrow \qquad x^2 = 0$ or $x - 1 = 0$

$\Rightarrow \qquad x = 0$ or $x = 1$

Step #2: Now, we draw a sign table with the number line divided by the above *x* values.

Intervals	$(-\infty, 0)$	$(0,1)$	$(1, \infty)$
Test Value	$x = -1$	$x = 0.5$	$x = 2$
Sign of $\boldsymbol{x^2}$	$+ve$	$+ve$	$+ve$
Sign of $\boldsymbol{x - 1}$	$-ve$	$-ve$	$+ve$
Sign of $\boldsymbol{f(x) = 3x^2(x - 1)}$	$-ve$	$-ve$	$+ve$

As the inequality $3x^2(x-1) \geq 0$ includes when $3x^2(x-1) > 0$ and $3x^2(x-1) = 0$, therefore the solution to the inequality $3x^2(x-1) \geq 0$ is $x > 1$ or $x = 1$. Therefore, the overall solution is $x \geq 1$ and in interval notation, we can write that

∴ The solution to the inequality $3x^2(x-1) \geq 0$, or the domain of $\sqrt{3x^2(x-1)}$, is $x \geq 1$ or $x \in [1, \infty)$

Rational Inequalities

The method to solve a rational inequality is similar to the method of solving a polynomial inequality, i.e. we utilise a sign table based on intervals on the number line.

It is **important** to remember that in a rational, the denominator must never be zero.

Example: Solve the inequality $\frac{x^2-1}{x} \leq 0$

Let $f(x) = \frac{x^2-1}{x} = \frac{(x-1)(x+1)}{x}$

Step #1: Assume that $(x^2 - 1)(x) = 0$.

(Notice that: We assume that all the factors in the numerator and the denominator multiply.)

$\Rightarrow \quad (x-1)(x+1)(x) = 0$

$\Rightarrow \quad x - 1 = 0$ or $x + 1 = 0$ or $x = 0$

$\Rightarrow \quad x = 1$ or $x = -1$ or $x = 0$

Step #3: Now, we draw a sign table with the number line divided by the above *x* values.

Intervals	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,\infty)$
Test Value	$x=-2$	$x=-0.5$	$x=0.5$	$x=2$
Sign of $x-1$	$-ve$	$-ve$	$-ve$	$+ve$
Sign of $x+1$	$-ve$	$+ve$	$+ve$	$+ve$
Sign of x	$-ve$	$-ve$	$+ve$	$+ve$
Sign of $f(x)=\frac{(x-1)(x+1)}{x}$	$-ve$	$+ve$	$-ve$	$+ve$

At first glance, we may mistakenly write our solution to the inequality $\frac{(x-1)(x+1)}{x}\le 0$ as

$$x\in(-\infty,-1]\cup[0,1]$$

However, we must remember that the denominator can never be zero, i.e. $x\neq 0$. Therefore, the correct solution to the inequality $\frac{(x-1)(x+1)}{x}\le 0$ is

$$x\in(-\infty,-1]\cup(0,1]$$

Absolute Value Inequalities

If $c>0$, then

1. $|x|<c \quad \Leftrightarrow \quad -c<x<c$

   $|x|\le c \quad \Leftrightarrow \quad -c\le x\le c$

2. $|x|>c \quad \Leftrightarrow \quad x<-c$ or $x>c$

   $|x|\ge c \quad \Leftrightarrow \quad x\le -c$ or $x\ge c$

Examples:

1. $|x-5|<2$

$\Rightarrow \quad -2<x-5<2$

$\Rightarrow \quad -2+5<x<2+5$

$\Rightarrow \quad 3<x<7$ or $x\in(3,7)$

2. $|x+3|\ge 7$

$\Rightarrow \quad x+3\le -7$ or $x+3\ge 7$

$\Rightarrow \qquad x \leq -7 - 3$ or $x \geq 7 - 3$

$\Rightarrow \qquad x \leq -10$ or $x \geq 4$ OR $x \in (-\infty, -10] \cup [4, \infty)$

Modelling with Inequalities

Examples:

1. You go to a candy store to buy chocolates that cost \$9.89 per pound. The scale that is used in the store has a state seal of approval that indicates the scale is accurate to within $\frac{1}{32}$ of a pound. According to the scale, your purchase weighs one-half pound and costs \$4.95. How much might you have been undercharged or overcharged as a result of inaccuracy in the scale?

Solution: You bought $\frac{1}{2}$ lb of chocolates but let *x* be the actual weight of your chocolates. The scale is accurate to plus or minus $\frac{1}{32}$ lb. So we can say that

$$\frac{1}{2} - \frac{1}{32} \leq x \leq \frac{1}{2} + \frac{1}{32}$$

$$\Rightarrow \qquad \frac{16}{32} - \frac{1}{32} \leq x \leq \frac{16}{32} + \frac{1}{32}$$

$$\Rightarrow \qquad \frac{15}{32} \leq x \leq \frac{17}{32}$$

The chocolates cost \$9.89 per pound, so the actual cost of the chocolates should be

$$\$9.89 \times \frac{15}{32} \leq \$9.89x \leq \$9.89 \times \frac{17}{32}$$

$$\Rightarrow \qquad \$9.89 \times \frac{15}{32} \leq \$9.89x \leq \$9.89 \times \frac{17}{32}$$

$$\Rightarrow \qquad \$4.64 \leq \$9.89x \leq \$5.25$$

Since you paid \$4.95 for the chocolates, you were either undercharged by as much as

$$\$5.25 - \$4.95 = \$0.30$$

or you were overcharged by as much as

$$\$4.95 - \$4.64 = \$0.31$$

2. You are considering two job offers. The first job pays \$3000 per month. The second job pays \$1000 per month plus a commission of 4% of your gross sales. Write an inequality yielding the gross sales per month for which the second job will pay the greater monthly wage. Solve the inequality.

Solution: Let *x* be the gross sales per month.

The first job pays \$3000 per month.

The second job pays \$1000+ 4% of *x* $= \$1000 + 0.04x$

We want to know what is *x* when

$$\$1000 + 0.04x > \$3000$$

$\Rightarrow \quad 0.04x > \$3000 - \$1000$

$\Rightarrow \quad 0.04x > \$2000$

$\Rightarrow \quad x > \frac{\$2000}{0.04}$

$\Rightarrow \quad x > \frac{\$2000}{\frac{4}{100}}$

$\Rightarrow \quad x > \frac{\$2000}{\frac{1}{25}}$

$\Rightarrow \quad x > \$2000 \times 25$

$\Rightarrow \quad x > \$50{,}000$

For the second job to pay the greater monthly wage, the gross sales must exceed \$50,000 per month.

3. In order for an investment of \$750 to grow to more than \$825 in 2 years, what must the annual interest rate be? (Formula: $A = P(1 + rt)$)

Solution: Given $P = \$750$ and $t = 2$.

We want $A > \$825$

$\Rightarrow \quad P(1 + rt) > \$825$

$\Rightarrow \quad \$750(1 + r \times 2) > \$825$

$\Rightarrow \quad 1 + 2r > \frac{\$825}{\$750}$

$\Rightarrow \quad 1 + 2r > \frac{11\times75}{10\times75}$

$\Rightarrow \quad 1 + 2r > \frac{11}{10}$

$\Rightarrow \quad 2r > \frac{11}{10} - 1$

$\Rightarrow \quad 2r > \frac{11}{10} - \frac{10}{10}$

$\Rightarrow \quad 2r > \frac{1}{10}$

$\Rightarrow \quad r > \frac{1}{2\times 10}$

$\Rightarrow \quad r > \frac{1}{20}$

$\Rightarrow \quad r > 0.05$

In order for an investment of $750 to grow to more than $825 in 2 years, the annual interest rate must exceed 0.05 or 5%.

4. The revenue from selling *x* units of a product is $R = 115.95x$. The cost of producing *x* units is $C = 95x + 750$. To obtain a profit, the revenue must be greater than the cost. For what values of *x* will this product return a profit?

Solution: To obtain a profit, $R > C$
Or $115.95x > 95x + 750$

$\Rightarrow \quad 115.95x - 95x > 750$

$\Rightarrow \quad 20.95x > 750$

$\Rightarrow \quad x > \frac{750}{20.95}$

$\Rightarrow \quad x > 35.8$

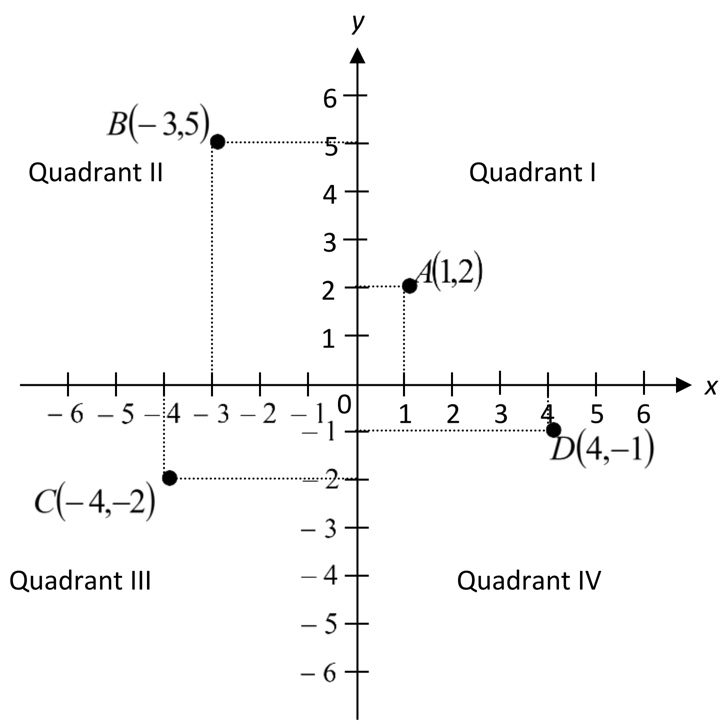
In order to return a profit, we must sell at least 36 units.

1.8 Coordinate Geometry

The Coordinate Plane

The rectangular coordinate system is also known as the Cartesian Coordinate System named after the French mathematician Rene Descartes (1596 – 1650).

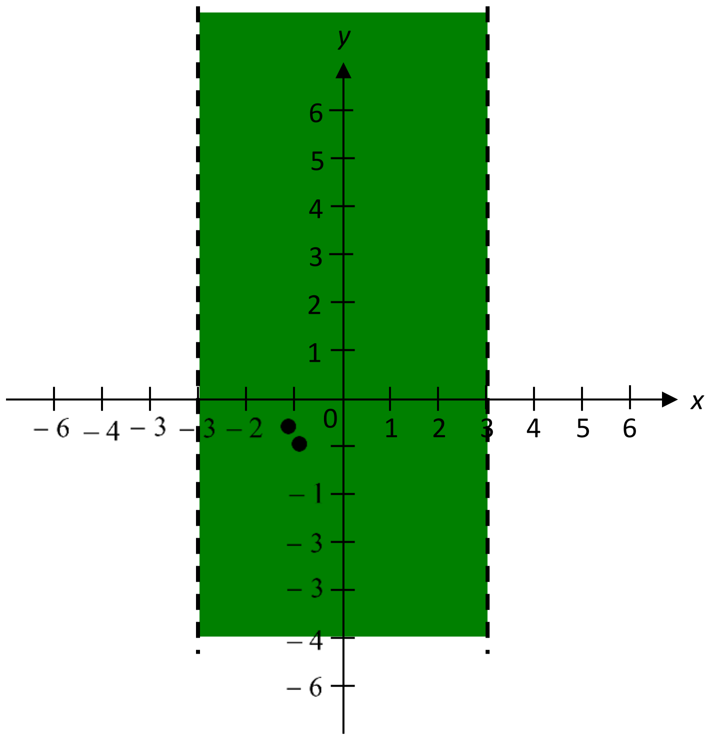
- Consists of a pair of real numbers written as (x, y).
- Used to pinpoint a specific position on an *xy*-plane as shown below:



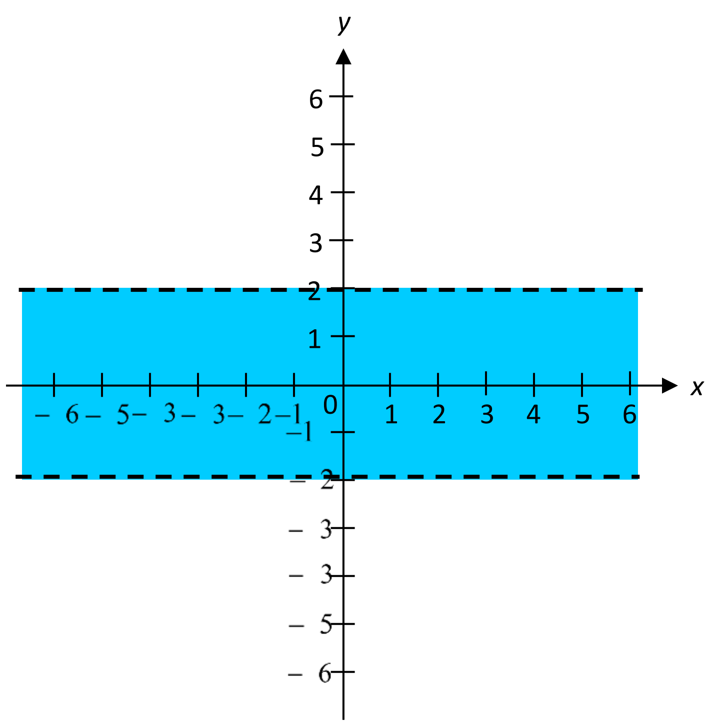
Example: Sketch the region given by the set $\{(x, y)\ /\ |x| < 3\ \ and\ |y| < 2\}$.

There are two inequalities here. We need to look at both of them separately and then see which region is common to both inequalities, or which region is an overlapping of both inequalities.

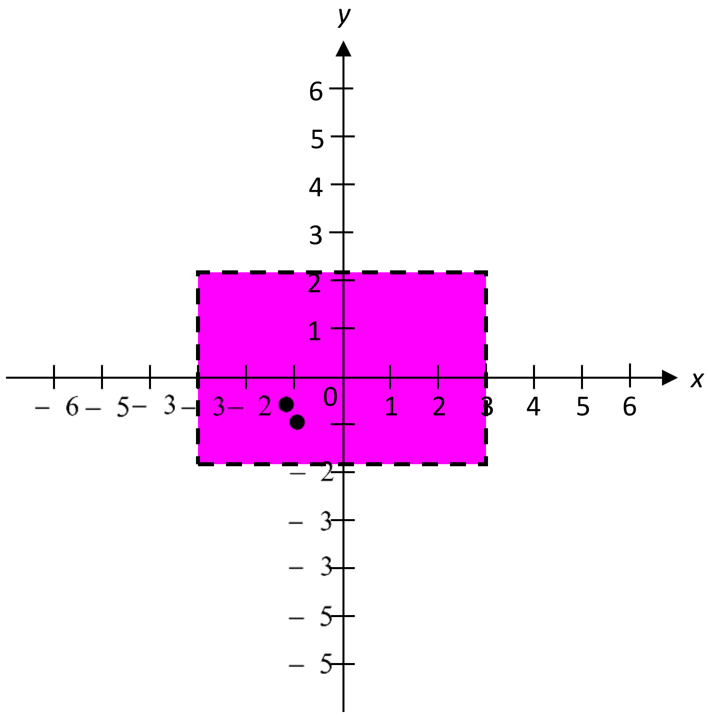
The first inequality is $|x| < 3$ or $-3 < x < 3$. So the region where $-3 < x < 3$ is



Next, we look at the second inequality $|y| < 2$ or $-2 < y < 2$. This region is



Therefore, the overlapping region where $|x| < 3$ and $|y| < 2$ is

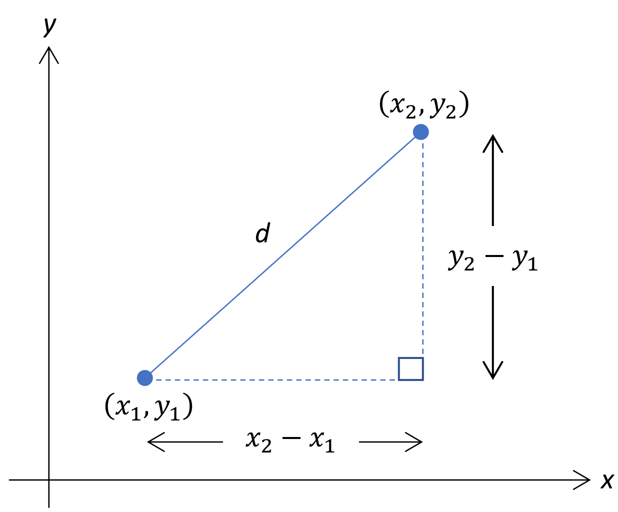


The Distance Formula

Suppose we have two points, (x_1, y_1) and (x_2, y_2), on the Cartesian plane as shown below.

The horizontal distance, also known as ***run***, between the two points is $x_2 - x_1$.

The vertical distance, also known as ***rise***, between the two points is $y_2 - y_1$.



We can use the Pythagorean Theorem to find the distance between these two points:

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = d^2$$

$$\Rightarrow \qquad d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example:

Find the distance between two points $(-2,1)$ and $(3,4)$.

Solution: Let $(x_1, y_1) = (-2,1)$
And $(x_2, y_2) = (3,4)$

Thus, the distance between these two points is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{\left(3 - (-2)\right)^2 + (4 - 1)^2}$$

$$= \sqrt{(3 + 2)^2 + (4 - 1)^2}$$

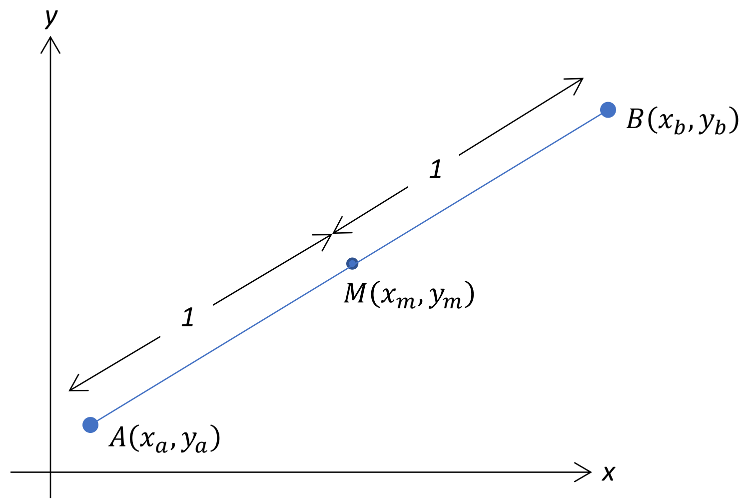
$$= \sqrt{5^2 + 3^2}$$

$$= \sqrt{25 + 9}$$

$$= \sqrt{34} \textit{ units}$$

The Midpoint Formula

Let us say we are given a straight line joining two points, $A(x_a, y_a)$ and $B(x_b, y_b)$, as shown below. Now suppose we wish to find the midpoint of these two points.



Notice that the midpoint, $M(x_m, y_m)$, of the line joining two points is the point that divides this enjoining line in the ratio of $1{:}\,1$. Thus, the midpoint formula is obtained from the above by setting the ratio $m{:}\,n$ as $1{:}\,1$.

So we have,

$$M = \frac{1 \times A + 1 \times B}{1+1}$$

$$\Rightarrow \qquad M = \frac{A+B}{2}$$

This gives us the coordinates of the midpoint of the line segment joining two points, $A(x_a, y_a)$ and $B(x_b, y_b)$ as

$$M = \left(\frac{x_b + x_a}{2}, \frac{y_b + y_a}{2}\right)$$

Note: Notice that this is simply like finding the average values of the *x* and the *y* coordinates of the two endpoints.

Example: Find the midpoint of the line segment joining points $(-5,-3)$ and $(9,3)$.

Solution: Let the midpoint be $M(x_m, y_m)$. The midpoint formula states that the midpoint is given by

$$M = \left(\frac{x_b+x_a}{2}, \frac{y_b+y_a}{2}\right)$$

$$= \left(\frac{-5+9}{2}, \frac{-3+3}{2}\right)$$

$$\left(\frac{4}{2}, \frac{0}{2}\right) = (2,0)$$

Intercepts

[1] The *x*-intercepts are the points where the graph of an equation intersects the *x*=axis.

To find the *x*-intercepts: Set $y = 0$ and solve for *x*.

[2] The *y*-intercepts are the points where the graph of an equation intersects the *y*-axis.

To find the *y*-intercepts: Set $x = 0$ and solve for *y*.

Note: In a function, there can be as many *x*-intercepts as possible, but there can only be one *y*-intercept. Can you think of the reason why?

Example:

Find the axes intercepts of $y = x + \sqrt{2-x}$

First of all, we need to find the domain of $y = x + \sqrt{2-x}$.
We can see there is a term with a square root function and we need the argument of the square root to be non-negative, i.e. we need

$$2 - x \geq 0$$

$$\Rightarrow \qquad x \leq 2$$

To find the *x*-intercepts: Set $y = 0$ and solve for *x*.

$$y = x + \sqrt{2-x}$$

$$\Rightarrow \qquad 0 = x + \sqrt{2-x} \qquad \text{Setting } y = 0$$

$$\Rightarrow \qquad \sqrt{2-x} = -x$$

$$\Rightarrow \qquad 2 - x = (-x)^2 = x^2$$

$\Rightarrow \qquad x^2 + x - 2 = 0$

$\Rightarrow \qquad (x+2)(x-1) = 0$

$\Rightarrow \qquad x + 2 = 0$ or $x - 1 = 0$

$\Rightarrow \qquad x = -2$ or $x = 1$

Since both these numbers are on the domain, can we safely conclude that these are the *x*-intercepts? Not quite yet. We still have to check and confirm if these are indeed the x-intercepts. We do this by substituting our answers to the *x*-variable in the original equation

$$y = x + \sqrt{2 - x}$$

and we check if $y = 0$ is satisfied.

Let us substitute $x = -2$, we have

$$y = -2 + \sqrt{2 - (-2)}$$

$$= -2 + \sqrt{2 + 2}$$

$$= -2 + \sqrt{4}$$

$$= -2 + 2 = 0$$

We see that $y = 0$ when $x = -2$, so this confirms that $x = -2$ is indeed an *x*-intercept.

Now, let us substitute $x = 1$, we have

$$y = 1 + \sqrt{2 - 1}$$

$$= 1 + \sqrt{1}$$

$$= 1 + 1 = 2$$

We see that $y \neq 0$ when $x = 1$, so this shows that $x = 1$ is not an *x*-intercept.

$\therefore$ In conclusion, $y = x + \sqrt{2 - x}$ has only one *x*-intercept, i.e. $x = -2$.

To find the *y*-intercept: Since $x = 0$ is on the domain, we can also safely set $x = 0$ and solve for *y*.

$$y = x + \sqrt{2 - x}$$

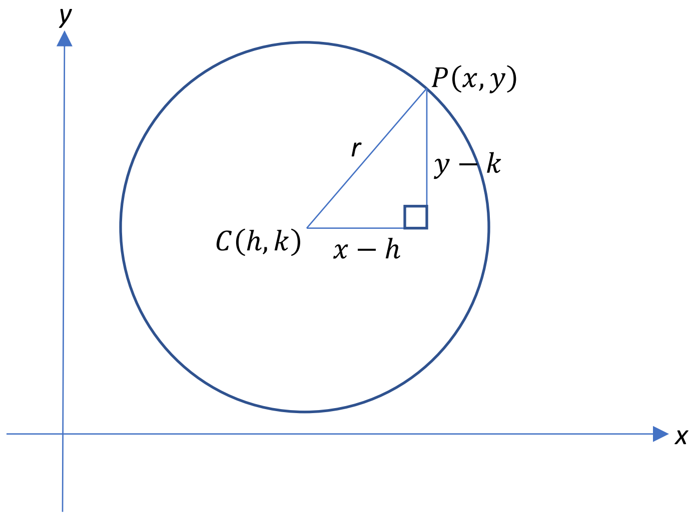
$\Rightarrow \qquad y = 0 + \sqrt{2 - 0} \qquad$ Setting $x = 0$

$$\Rightarrow \qquad y = \sqrt{2}$$

This is the *y*-intercept.

Circles

Let us consider a circle with centre $C(h, k)$ and radius of length *r*. Let $P(x, y)$ be any arbitrary point on the circle.



From the Pythagoras' Theorem, we can see that

$$\boxed{(x-h)^2 + (y-k)^2 = r^2}$$

This is the standard form for the equation of a circle with centre $C(h, k)$ and radius of length *r*.

Example: Find the equation of the circle where the endpoints of a diameter are $P(-1, 3)$ and $Q(7, -5)$.

Solution: The centre of the circle is the midpoint of $P(-1, 3)$ and $Q(7, -5)$.

Hence, $C = \left(\frac{-1+7}{2}, \frac{3+(-5)}{2}\right)$

$$\Rightarrow \qquad (h, k) = \left(\frac{6}{2}, \frac{-2}{2}\right) = (3, -1)$$

Its diameter is the distance between $P(-1, 3)$ and $Q(7, -5)$

$$d = \sqrt{\left(7-(-1)\right)^2 + (-5-3)^2}$$

$$= \sqrt{8^2 + (-8)^2} = \sqrt{64+64} = \sqrt{(2)(64)}$$

$$= \sqrt{64}\sqrt{2}$$

$$= 8\sqrt{2}\, units$$

The radius of a circle is half its diameter, hence

$$r = \frac{d}{2} = \frac{8\sqrt{2}}{2} = 4\sqrt{2}\, units$$

∴ The equation of the circle is

$$(x-h)^2 + (y-k)^2 = r^2$$

$$\Rightarrow \quad (x-3)^2 + \left(y-(-1)\right)^2 = \left(4\sqrt{2}\right)^2$$

$$\Rightarrow \quad (x-3)^2 + (y+1)^2 = (16)(2)$$

$$\Rightarrow \quad (x-3)^2 + (y+1)^2 = 32$$

Check: Let's plug in the two points given in the question:

Given $P(-1, 3)$, i.e. $x = -1$ and $y = 3$. So we have

$$(-1-3)^2 + (3+1)^2 = (-4)^2 + 4^2 = 16 + 16 = 32$$

Given also $Q(7, -5)$, i.e. $x = 7$ and $y = -5$. So we have

$$(7-3)^2 + (-5+1)^2 = 4^2 + (-4)^2 = 16 + 16 = 32$$

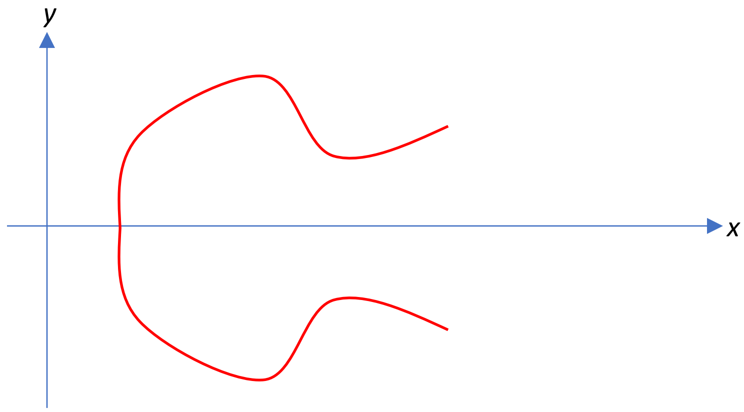
Symmetry

In this topic, we shall discuss three types of symmetry:

[1] Symmetry about the *x*-axis.

The *x*-axis is like a mirror, the parts of the graph above and below the *x*-axis are mirror images of each other.

Test: The equation is unchanged when we replace *y* by $-y$.

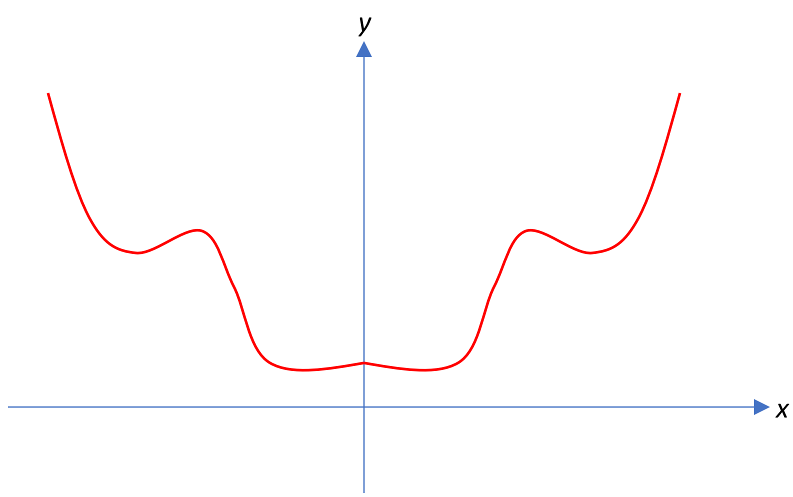


Note: A graph with this property is the graph of a non-function.

[2] Symmetry about the *y*-axis.

The *y*-axis is like a mirror, the parts of the graph to the left and to the right of the *y*-axis are mirror images of each other.

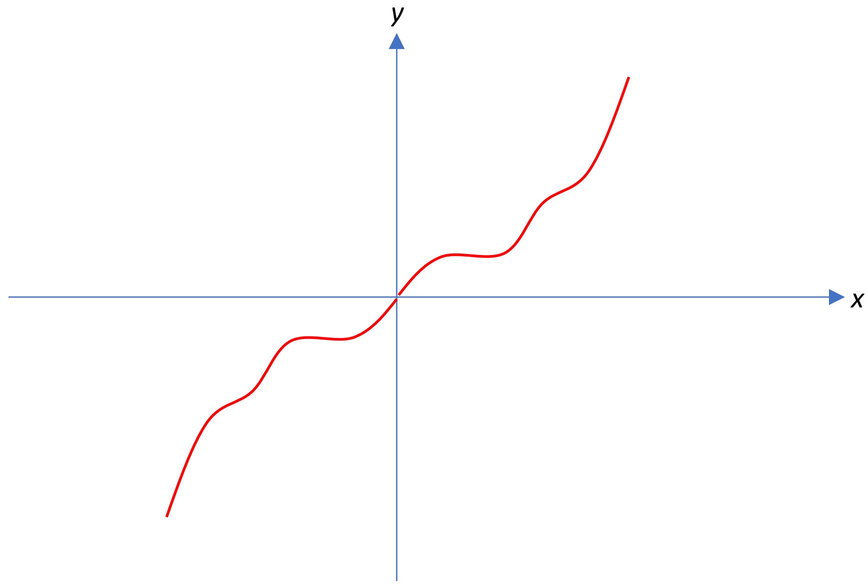
Test: *y* is unchanged when we replace *x* by $-x$.



[3] Symmetry about the origin $O(0, 0)$

Imagine we pin a tack at the origin $O(0, 0)$ and we turn the graph upside down. If the graph maps back onto itself after the turn, then we can safely say this graph is symmetric about the origin $O(0, 0)$.

Test: When we replace *x* by $-x$, the result is that *y* is replaced by $-y$.



Note: Only a small handful of graphs have symmetry. The majority of graphs have no symmetry whatsoever.

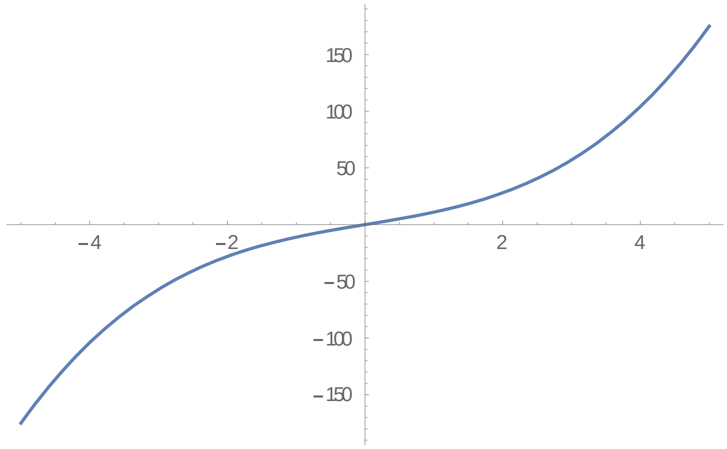
Examples: Test the equations for symmetry:

[1] $y = x^3 + 10x$

Replace *x* by $-x$: $y = (-x)^3 + 10(-x) = -x^3 - 10x = -(x^3 + 10x) = -y$

When we replace *x* by $-x$, the result is that *y* is replaced by $-y$. This indicates that the graph of $y = x^3 + 10x$ is symmetric about the origin $O(0, 0)$.

The graph of $= x^3 + 10x$:

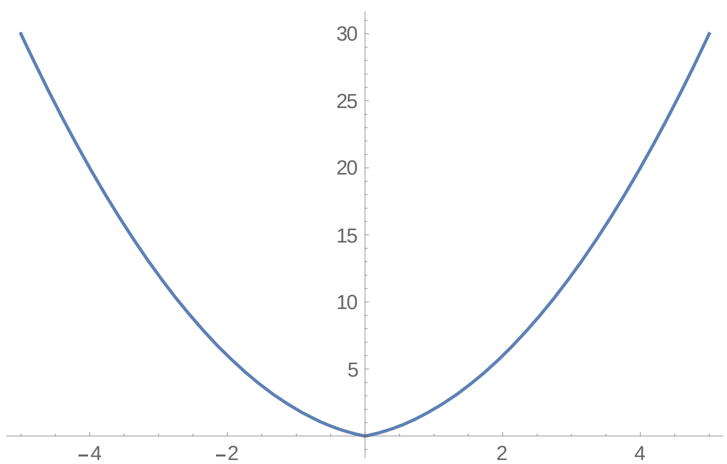


[2] $y = x^2 + |x|$

Replace *x* by $-x$: $y = (-x)^2 + |-x| = x^2 + |x| = y$

We can see that *y* remains unchanged when we replace *x* by $-x$. This indicates that the graph of $y = x^2 + |x|$ is symmetric about the *y*-axis.

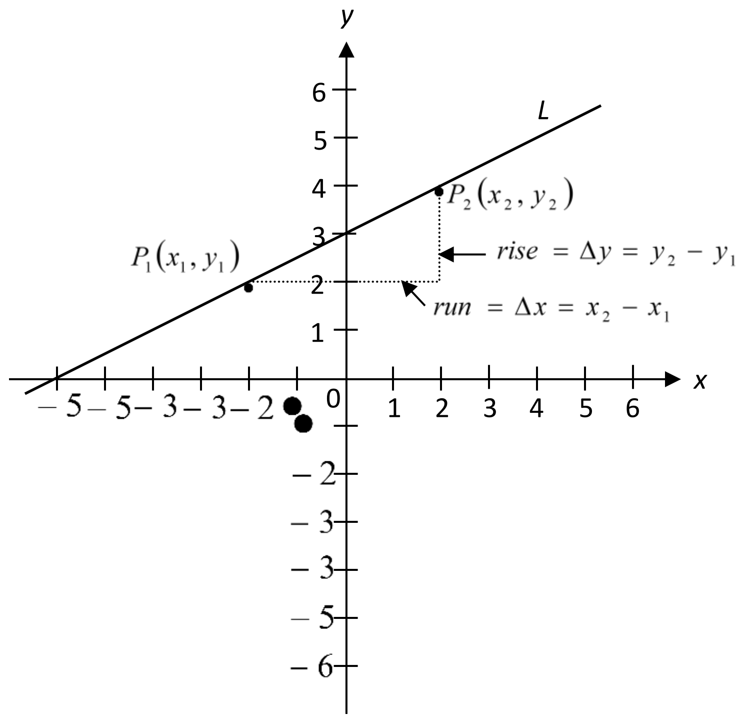
Graph of $y = x^2 + |x|$:



1.9 Lines

The Slope of a Line

Let us consider a straight line *L* passing through two points, i.e. $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.



The slope of a non-vertical line that passes through $P_1(x_1, y_1)$ and $P_1(x_1, y_1)$ is

$$\boxed{m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1}}$$

Examples:

Find the slope of the line through *P* and *Q*.

1. $P(-1,6)$, $Q(4,-3)$

Slope, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{-3-6}{4-(-1)} = \frac{-9}{5}$$

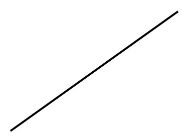
2. $P(-1,-4)$, $Q(6,0)$

Slope, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

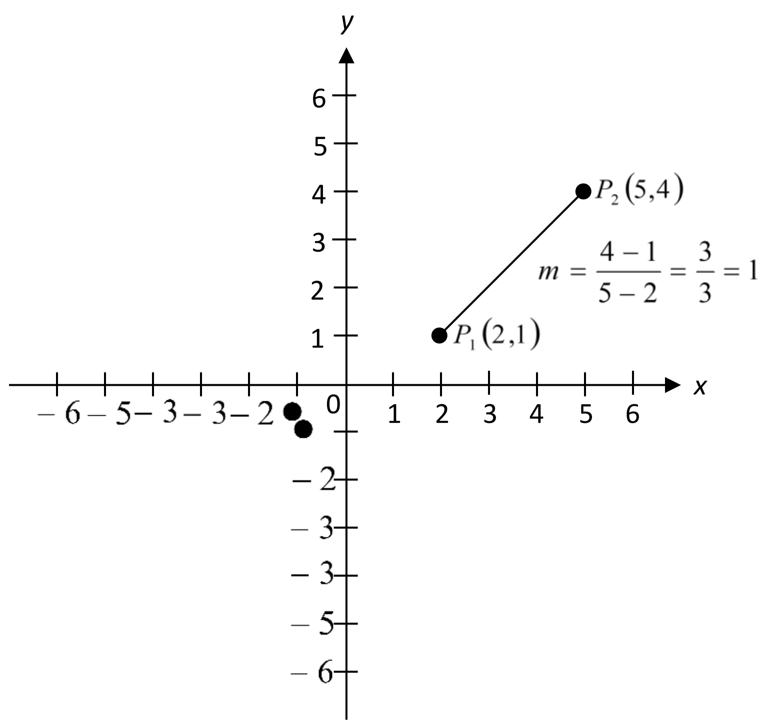
$$= \frac{0-(-4)}{6-(-1)} = \frac{4}{7}$$

Observations

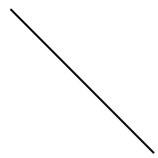
Notice that if a line is in the bottom left to the top right direction, i.e.



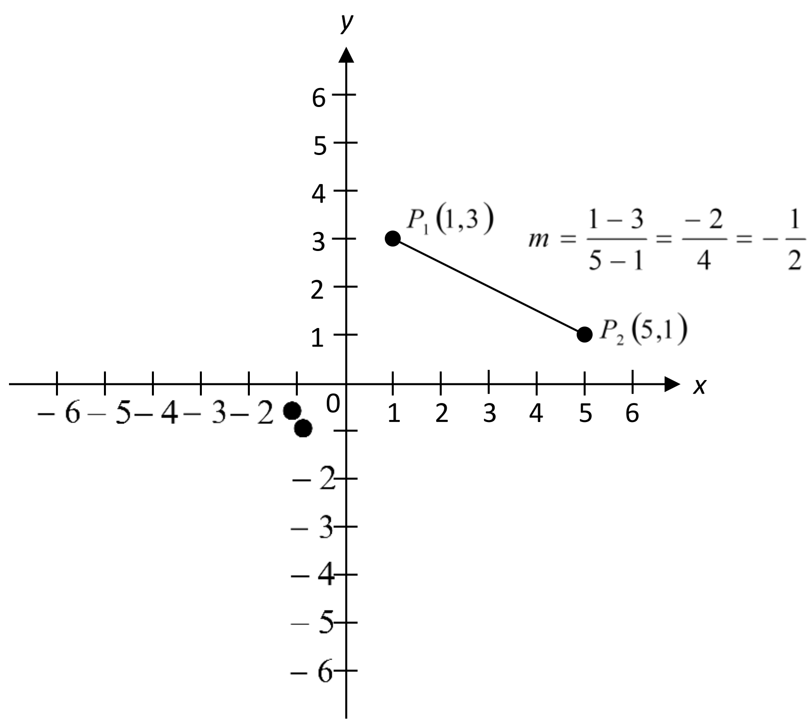
its slope takes a *+ve* value. For example,



If a line is in the top left to bottom right direction, i.e.

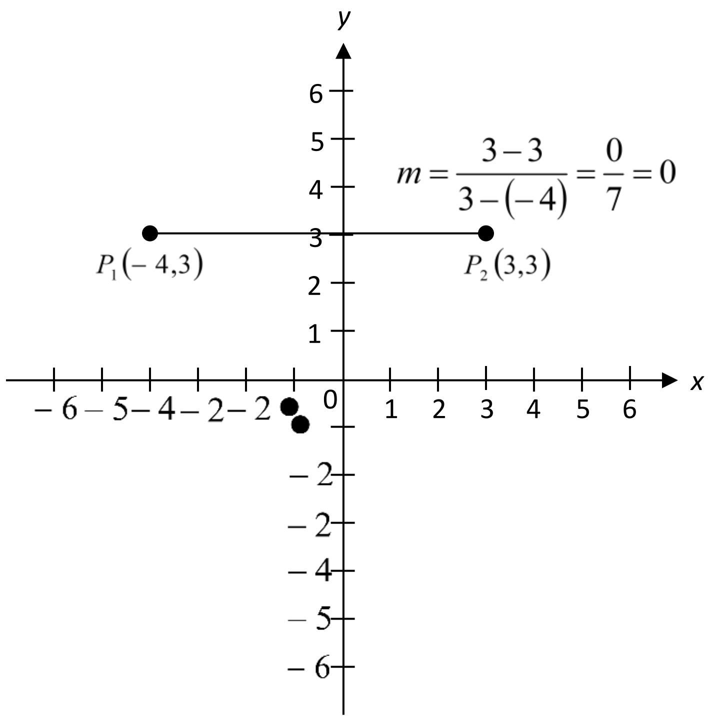


its slope takes a *–ve* value. For example,



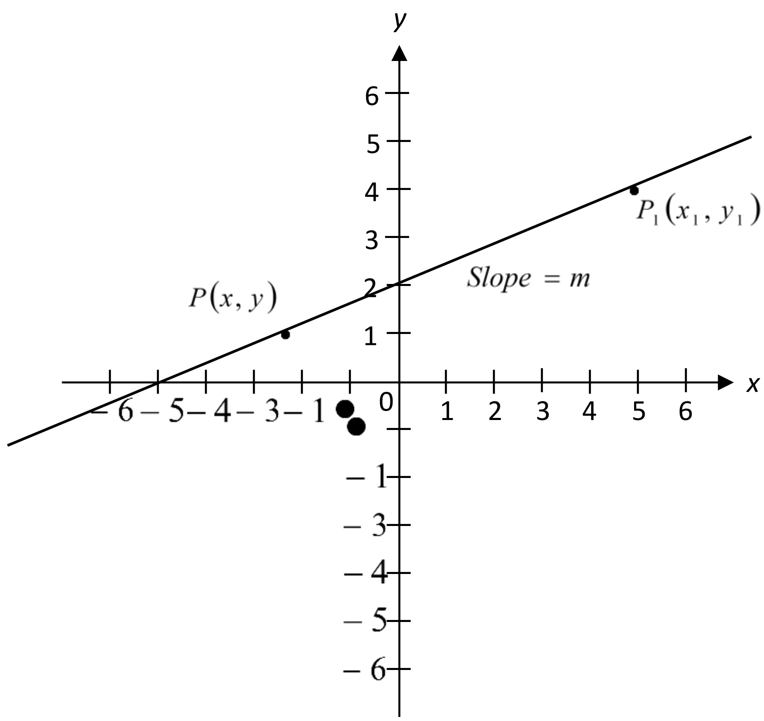
And if a line is horizontal, i.e.

Its slope takes a zero value. For example,



Point-Slope Form of the Equation of a Line

Let us suppose that there is a general point $P(x, y)$ that lies anywhere on a non-vertical line. Let us also suppose that there is a specific point $P_1(x_1, y_1)$ that lies on the same line. Let us further suppose that the slope of the line is *m*. What happens now?



We know that the slope of the line can be calculated from

$$m = \frac{y - y_1}{x - x_1}$$; Slope of line joining points $P(x, y)$ and $P_1(x_1, y_1)$

From here, we can obtain the **Point-Slope Form of the Equation of a Line** :

$$y - y_1 = m(x - x_1)$$

Note: To find the equation of a straight line, we need to know two things:

1. The slope of the line, *m*.
2. Any one point that lies on the line, (x_1, y_1)

Slope-Intercept Form of the Equation of a Line

We can always convert the Point-Slope Form to another form called the Slope-Intercept Form.

$$y - y_1 = m(x - x_1)$$
$$\Rightarrow \quad y - y_1 = mx - mx_1$$
$$\Rightarrow \quad y = mx + y_1 - mx_1$$

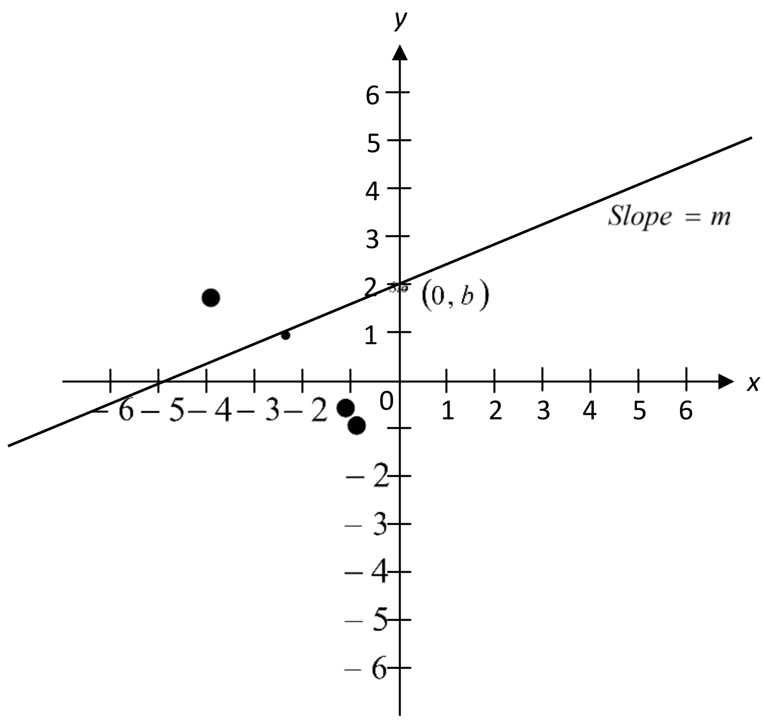
We let $b = y_1 - mx_1$ and we can rewrite the above as

$$y = mx + b$$

Notice that when $x = 0$, we have

$$y = b$$

which is the *y*-intercept, or where the straight line intersects the *y*-axis.



Example: Find an equation of the line that passes through $(-1,4)$ with slope -3.

Answer: Here, the line passes through $(x_1, y_1) = (-1,4)$. Therefore, the equation of the line is

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow \quad y - 4 = -3(x - (-1))$$

$$= -3(x + 1)$$

$$= -3x - 3$$

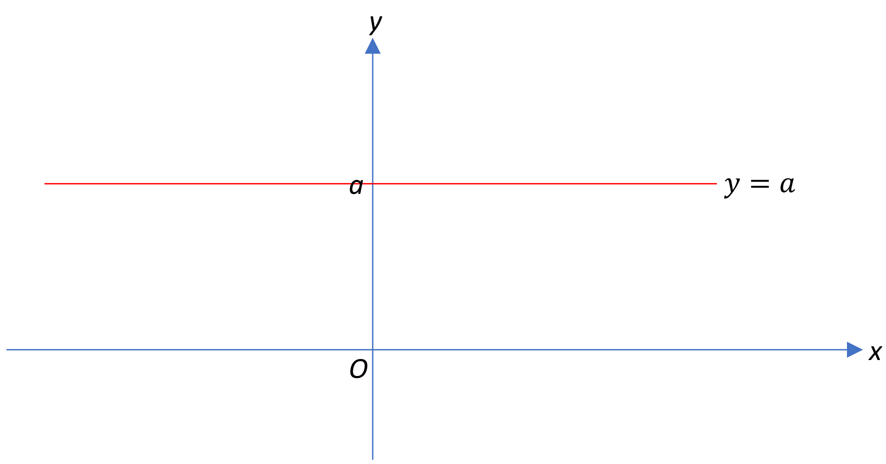
$$\Rightarrow \quad y = -3x - 3 + 4$$

$$\Rightarrow \quad y = -3x + 1$$

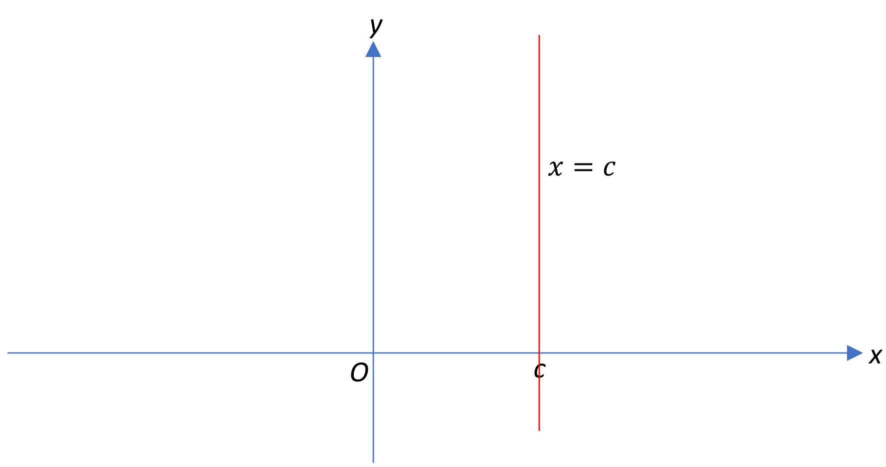
And we can conclude that the *y*-intercept of this line is at $(0, 1)$.

Vertical and Horizontal Lines

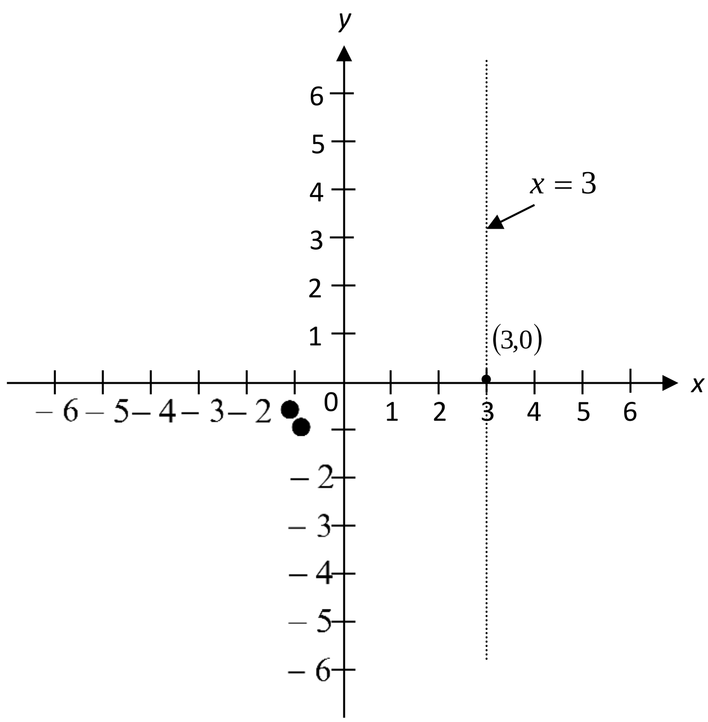
A horizontal line that intersects the *y*-axis at $(0, a)$ is simply written as $y = a$.



A vertical line that intersects the *x*-axis at $(c, 0)$ is written as $x = c$.



For example, if a line passes through $(3,0)$,



General First-Degree Equation of a Line

Every straight line can be written in the form

$$Ax + By + C = 0$$

$$\Rightarrow \qquad By = -Ax - C$$

$$\Rightarrow \qquad y = -\frac{A}{B}x - \frac{C}{B}$$

$\therefore$ The slope is $m = -\dfrac{A}{B}$ and the *y*-intercept is $b = -\dfrac{C}{B}$.

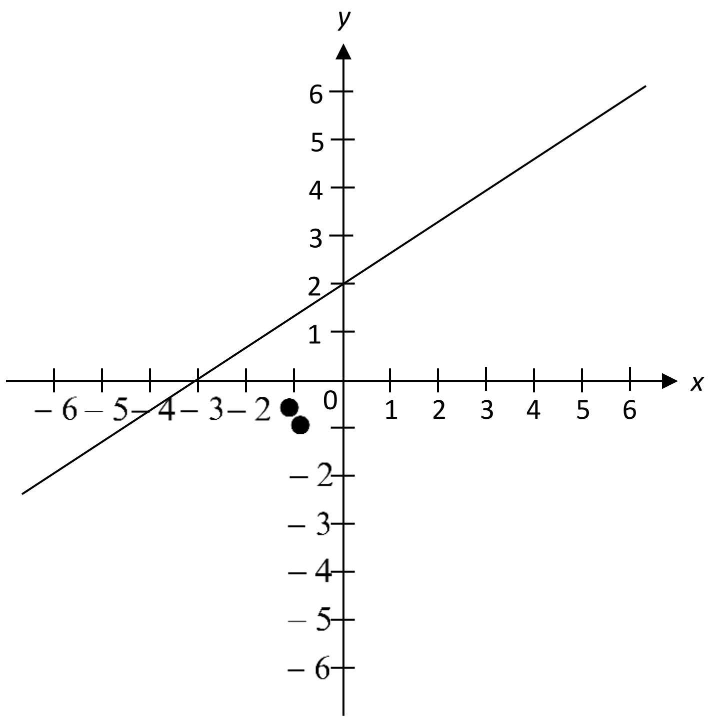
Example:

Find the slope and the *y*-intercept of $2x - 3y + 6 = 0$ and draw its graph.

Answers: From the line equation, $A = 2$, $B = -3$ and $C = 6$. Therefore, we have

Slope, $m = -\dfrac{A}{B} = -\dfrac{2}{-3} = \dfrac{2}{3}$

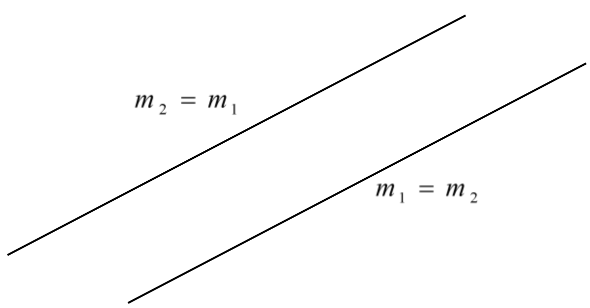
and y-intercept, $b = -\dfrac{C}{B} = -\dfrac{6}{-3} = 2$



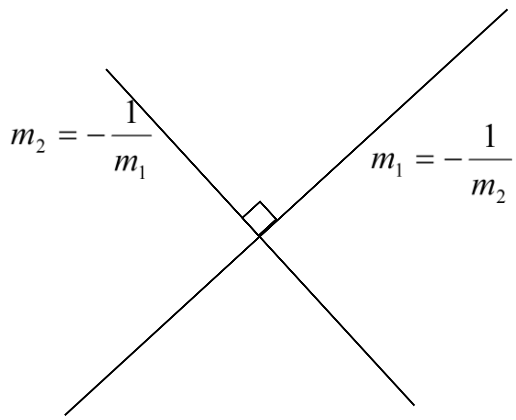
Parallel and Perpendicular Lines

Properties:

1. Two non-vertical lines are parallel if and only if they have the same slope.



2. Two lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$, or $m_2 = -\frac{1}{m_1}$, or $m_1 = -\frac{1}{m_2}$.



Application Example

Given a triangle with vertices $A(6,-7)$, $B(11,-3)$ and $C(2,-2)$. Use slopes to show that *ABC* is a right triangle.

Slope of $AB = \frac{-3-(-7)}{11-6} = \frac{4}{5}$

Slope of $AC = \frac{-2-(-7)}{2-6} = \frac{5}{-4} = -\frac{5}{4}$

Since $(\text{Slope of } AB) \times (\text{Slope of } AC) = -1$,

$\Rightarrow$ *AB* and *AC* are perpendicular with each other.

$\therefore$ ΔABC is a right triangle with the right angle at vertex *A*.

Modelling with Linear Equations: Slope as Rate of Change

Example: When leather belts sell for \$4.00 each at the flea market, handcrafters offer 200 belts for sale each day. For each increase of \$0.10 in the selling price, another 5 belts are supplied to the market. Find the supply function as a linear function of price in terms of the number of belts supplied.

Solution: Let p be the price of each belt,
And Let q be the quantity of belts supplied.

Let the supply function be $p = mq + b$

We are told that when the price of each belt is \$4.00, handcrafters offer 200 belts for sale each day. This tells us that a point that lies on this straight line is

$$(q_1, p_1) = (200, 4)$$

We are also told that each increase of \$0.10 in the selling price results in another 5 belts being supplied. Since price is denoted by the vertical axis and quantity is denoted by the horizontal axis, we have

$$rise = 0.1 \qquad \text{and} \qquad run = 5$$

And the slope is $m = \frac{rise}{run} = \frac{0.1}{5} = \frac{1}{50}$

∴ The supply function is

$$p - p_1 = m(q - q_1)$$

$$\Rightarrow \quad p - 4 = \frac{1}{50}(q - 200)$$
$$= \frac{1}{50}q - \frac{200}{50}$$

$$\Rightarrow \quad p - 4 = \frac{1}{50}q - 4$$

$$\Rightarrow \quad p = \frac{1}{50}q$$

1.10 Making Models Using Variation

Direct Variation

$$y = kx$$

Here, *y* is directly proportional to *x*, i.e. as *x* increases do does *y* increase.

Examples include
- Income tax payable to income earned.
- Interest earned to amount of money in savings account.
- Brightness from a light bulb to amount of voltage supplied to the light bulb.

Example

When converting between inches and centimetres, it is defined that 13 inches is the same length as 33.02 centimetres. Find the number of centimetres in 10 inches.

Solution: Since the length in centimetres is directly proportional to its equivalent in inches, we have

$$y = kx$$

Where *y* is length in centimetres
And *x* its equivalent length in inches

We are given that when $x = 13$ *in*, $y = 33.02$ *cm*

Thus, $33.02 = k \times 13$

$\Rightarrow \quad k = \frac{33.02}{13}$

$\therefore$ The model is $y = \frac{127}{50}x = 2.54x$

When $x = 10$ *in*, its equivalent in centimetres is

$$y \approx 2.54 \times 10 \approx 25.4 \text{ cm}$$

Something interesting: On the 1st of July 1959, the International Yard and Pound agreement was signed whereby it was agreed that $1\ yard = 0.9144\ m = 91.44\ cm$ exactly. This means that today, we can convert 1 *inch* as exactly 2.54 *cm*. The same agreement also defined $1\ lb = 0.453\ 592\ 37\ kg$ exactly.

Inverse Variation

$$y = \frac{k}{x}$$

Here, y is inversely proportional to x, i.e. if x increases, then y decreases. And if x decreases, then y increases.

Example:

On a particular construction project, it takes 20 people 10 hours to complete a job. How many hours will it take 25 people to complete this project?

Solution: Let y be the number of hours it takes to complete a job and x the number of people working on the project.

This is an inverse proportion problem, i.e.

$y = \frac{k}{x}$

When $x = 20, y = 10$

Therefore, $10 = \frac{k}{20}$

$\Rightarrow \quad k = 10 \times 20 = 200$

$\therefore$ The model is $y = \frac{200}{x}$

When $x = 25,$

$y = \frac{200}{25} = 8$ hours

It takes 25 people 8 hours to complete the job.

Joint Variation

Usually relationships between three or more variables are common.

For example,

$$z = k\frac{x}{y}$$

In the above example, z is directly proportional to x and inversely proportional to y.

Example: A moon that has a mass of 3×10^8 *kg* orbiting at a distance of 2×10^6 *m* away from the centre of a particular planet feels a tug of 7.5 *N*. What is the force felt by another moon 4×10^6 *kg* orbiting at a distance of 1×10^7 *m* away from the centre of this same planet?

Solution: We know from Newton's Law of Universal Gravitation that a force felt by an object away from another object is directly proportional to the mass of the first object and inversely proportional to the square of the distance separating the two objects. Thus, our model can be written as

$$F = k\frac{m}{r^2}$$

Where *k* is a constant coefficient,
m is the mass of the first object
And *r* is the distance between the two objects.

Given that $F = 7.5\ N,\ m = 3 \times 10^8\ kg$ and $r = 2 \times 10^6\ m$

We have $$7.5 = k\frac{3\times10^8}{(2\times10^6)^2} = k\frac{3\times10^8}{2^2\times10^{6\times2}}$$

$$= k\frac{3\times10^8}{4\times10^{12}}$$

$$= k \times 0.75 \times 10^{8-12}$$

$$= k \times 0.75 \times 10^{-4}$$

$$= k \times 7.5 \times 10^{-5}$$

$\Rightarrow$ $$k = \frac{7.5}{7.5\times10^{-5}} = 1 \times 10^5$$

$\therefore$ The model is $$F = 1 \times 10^5 \frac{m}{r^2}$$

The force felt by the other moon with mass $m = 4 \times 10^6\ kg$ orbiting at a distance of $r = 1 \times 10^7\ m$ is

$$F = 1 \times 10^5 \times \frac{4\times10^6}{(1\times10^7)^2}$$

$$= \frac{4\times10^{5+6}}{1^2\times10^{7\times2}} = \frac{4\times10^{11}}{1\times10^{14}}$$

$$= 4 \times 10^{11-14} = 4 \times 10^{-3}\ N$$

2.1 What is a Function?

Functions all around us

Examples:

[1] We go to the shop to buy apples. We select *x* number of apples and based on this number *x*, we have to pay *y* dollars for those apples.

[2] In school, we notice that the weight *y* of a school child is proportional to the child's height, *x*.

Definition

A function $f(x)$ is a relationship that assigns to each element *x* in a set, called its **domain**, exactly one element $f(x)$ in another set, called its **range**.

It is quite common to denote the variable *y* as the function value, i.e. $y = f(x)$

Note: $f(x)$ is read and understood as "*f* is a function in the variable *x*". It is not *f* multiplies *x*.

Examples:

[1] We go to the shop to buy apples. We select *x* number of apples and based on this number *x*, we have to pay *y* dollars for those apples.

Suppose each apple costs \$0.50. So we can write the function as

$$y = \$0.5x$$

[2] Converting the Fahrenheit scale to the Celcius scale:

$$C = \frac{5}{9}(F - 32)$$

Evaluating a Function

When evaluating a function $y = f(x)$, we replace the variable *x* with a value from its domain. And we replace every occurrence of *x* with that value.

Examples:

Suppose we are given the function

$$f(x) = 2x^2 + 3x - 1$$

We wish to evaluate the function at $x = -2$, $x = a$ and $x = a + h$

So we have $\quad f(x) = \quad 2x^2 \quad + \quad 3x \quad - 1$

$$\Rightarrow \quad f(-2) = 2(-2)^2 + 3(-2) - 1 = 1$$

If we replace *x* with -2, we replace every occurrence of *x* with -2

Also, $f(x) = 2x^2 + 3x - 1$

If we replace *x* with *a*, we replace every occurrence of *x* with *a*

$$\Rightarrow \quad f(a) = 2a^2 + 3a - 1$$

And $f(x) = 2x^2 + 3x - 1$

If we replace *x* with $a+h$, we replace every occurrence of *x* with $a+h$

$$\Rightarrow \quad f(a+h) = 2(a+h)^2 + 3(a+h) - 1$$

The Domain of a Function

The domain of a function, $f(x)$, is the set of all real numbers that we can possibly assign to the variable *x*.

Rules of thumb:

To find the domain, we remember the following rules:

- The argument of an even function must be non-negative.
- The denominator must not be zero.
- The argument of a logarithmic function must be positive.

Examples: Find the domain of the function:

[1] $f(x) = \frac{x+2}{x^2-1}$

We must ensure that the denominator must not be zero, i.e.

$$x^2 - 1 \neq 0$$

$$\Rightarrow \quad (x-1)(x+1) \neq 0$$

$$\Rightarrow \quad x - 1 \neq 0 \text{ and } x + 1 \neq 0$$

$$\Rightarrow \quad x \neq 1 \text{ and } x \neq -1$$

Hence, the domain of $f(x) = \frac{x+2}{x^2-1}$ is all real numbers except -1 and 1, i.e.

$$x < -1 \text{ or } -1 < x < 1 \text{ or } x > 1$$

OR

$$x \in (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$$

[2] $G(x) = \sqrt{x^2 - 9}$

We need the argument of an even root to be non-negative, i.e. here we need

$x^2 - 9 \geq 0$ To solve this, please refer to Lecture 1.7.

$\Rightarrow$ $x \leq -3$ or $x \geq 3$ This is the domain of $G(x) = \sqrt{x^2 - 9}$

OR $x \in (-\infty, -3] \cup [3, \infty)$

[3] $f(x) = \frac{x^2}{\sqrt{6-x}}$

We need the argument of an even root to be non-negative and the denominator to be non-zero, i.e.

$6 - x \geq 0$ and $6 - x \neq 0$

$\Rightarrow$ $x \leq 6$ and $x \neq 6$ Combining these two conditions give us

$\Rightarrow$ $x < 6$ This is the domain of $f(x) = \frac{x^2}{\sqrt{6-x}}$

OR $x \in (-\infty, 6)$

Common ways to represent a function

[1] Representing a function numerically

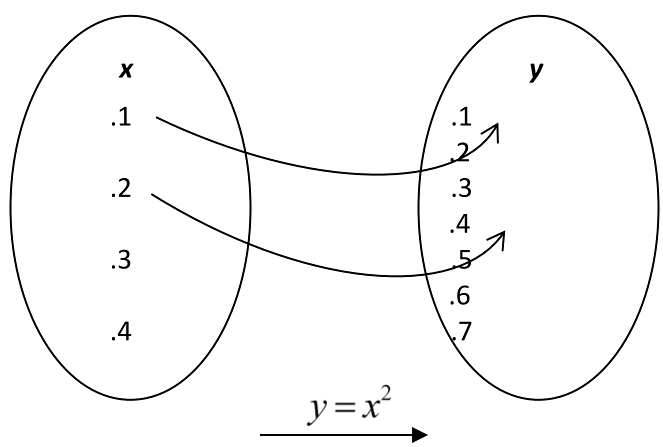
Example: World's Population

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080

[2] Representing a function visually

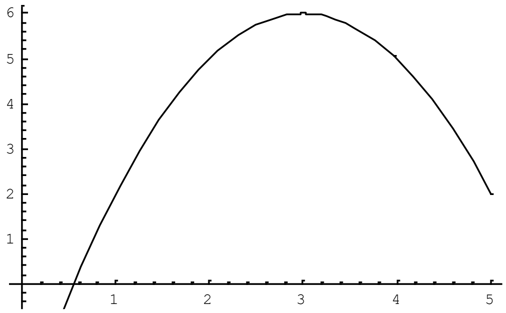
1. Arrow diagram. We have two sets of numbers. We use arrows to connect numbers from one set to numbers in another set based on what the function determines.

For example, $y = x^2$. The arrows from the left set of numbers point to their respective square values in the right set of numbers.



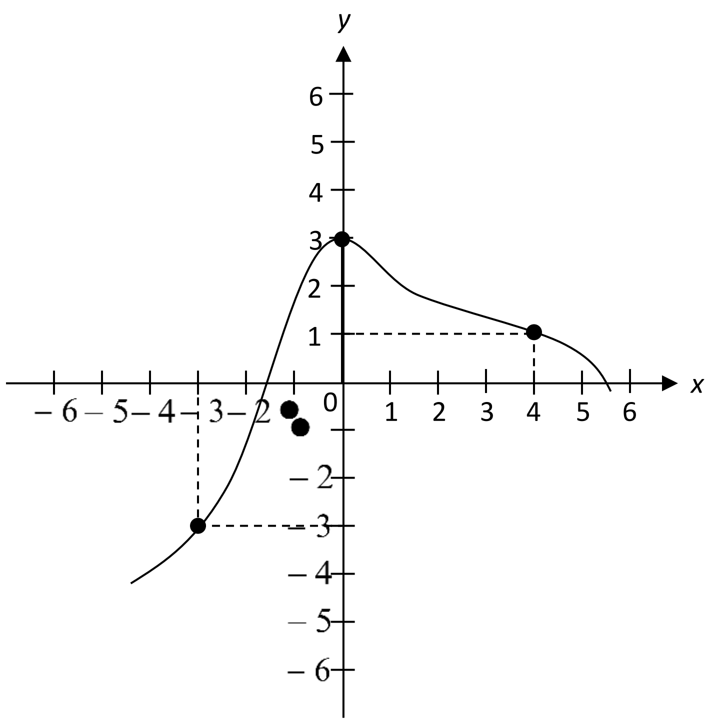
2. **Graphs.** We use the Cartesian Coordinate System where for each *x*-coordinate within the domain, the *y*-coordinate is determined by the value that the function returns.

Example: $y = -x^2 + 6x - 3$



Another example:

Suppose the graph below depicts a function $y = f(x)$.



We can write $f(-3) = -3$, $f(0) = 3$, $f(4) = 1$, ...

2.2 Graphs of Functions

Graphing Functions by Plotting Points

The graph of a function $y = f(x)$ is the set of ordered pairs (x, y) plotted on the Cartesian coordinate system.

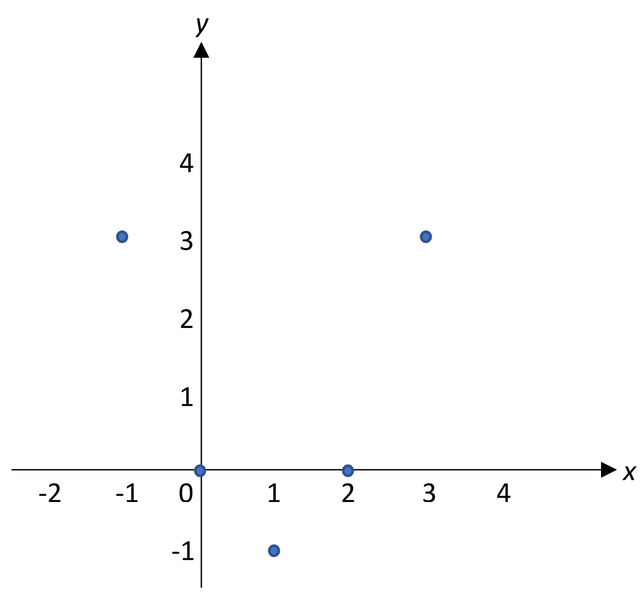
Example: Sketch the graph of the function using a table of ordered pairs.

[1] $g(x) = x^2 - 2x$

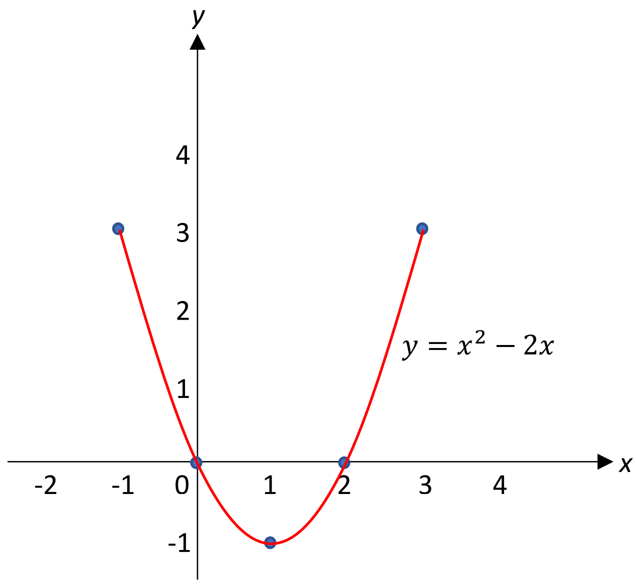
Step #1: We calculate the *y*-coordinates given the *x*-coordinates.

$\boldsymbol{x}$	$\boldsymbol{y = g(x)}$
-1	3
0	0
1	-1
2	0
3	3

Step #2: Then, we plot the points on the Cartesian coordinate system.



Step #3: Finally, we join the points with a smooth curve.



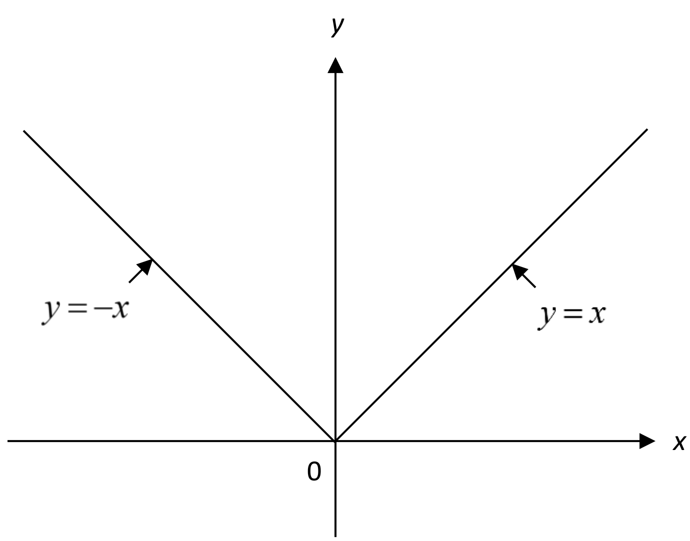
Graphing Piecewise Functions

A piecewise function comes in more than one piece, i.e. different formulas are defined on different parts of its domain.

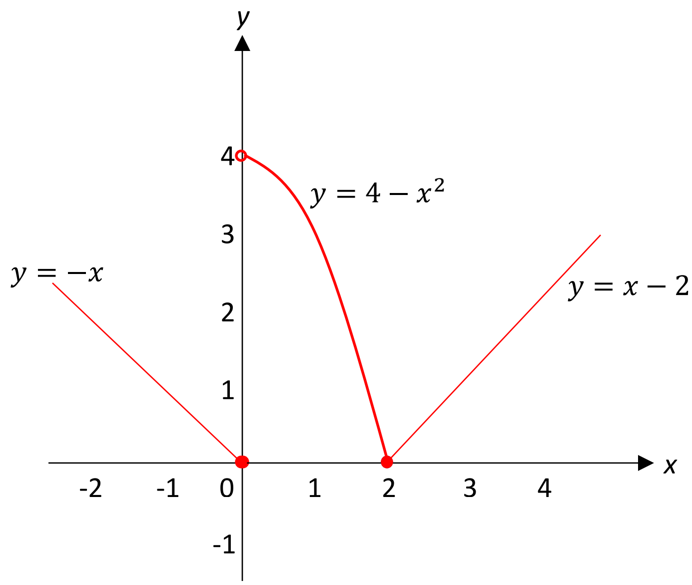
Examples:

1. The absolute value function,

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$



2. $$f(x) = \begin{cases} -x & if\ x \le 0 \\ 4 - x^2 & if\ 0 < x \le 2 \\ x - 2 & if\ x > 2 \end{cases}$$



An important property for a relationship to be defined as a function:

Every *x* in the domain may relate to one and only one *y*.

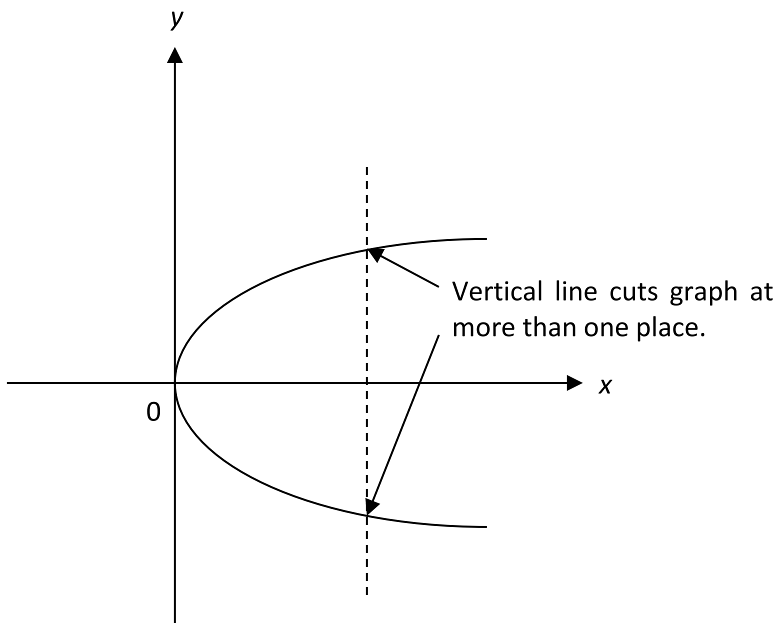
But the reverse is not necessary for a relationship to be defined as a function. A *y* may relate to more than one *x* in the domain.

So how do we test whether a relationship is a function or not? We perform what is known as the Vertical Line Test.

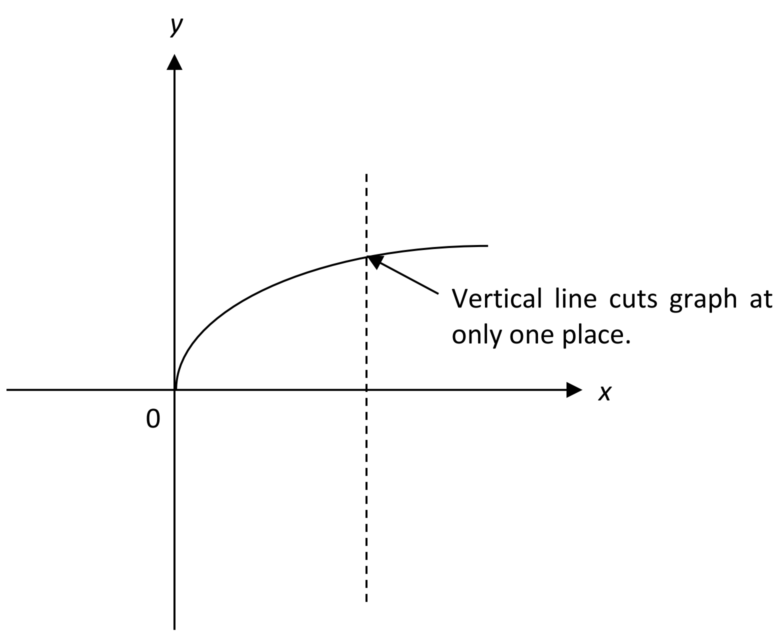
The Vertical Line Test: A curve in the *xy*-plane is the graph of a function of *x* if and only if no vertical line intersects the curve more than once.

Examples:

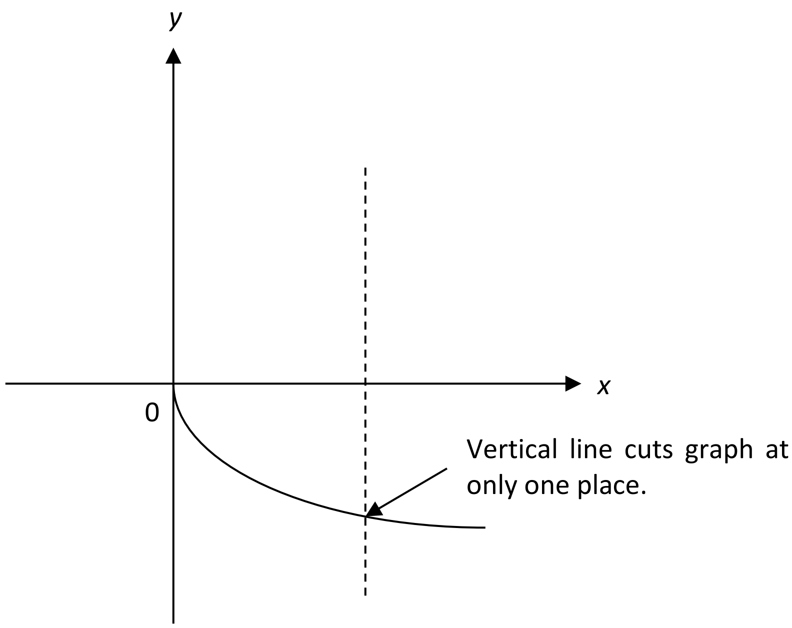
1. $y = \pm\sqrt{x}$ is NOT a function.



2. $y = \sqrt{x}$ is a function.



3. $y = -\sqrt{x}$ is a function.

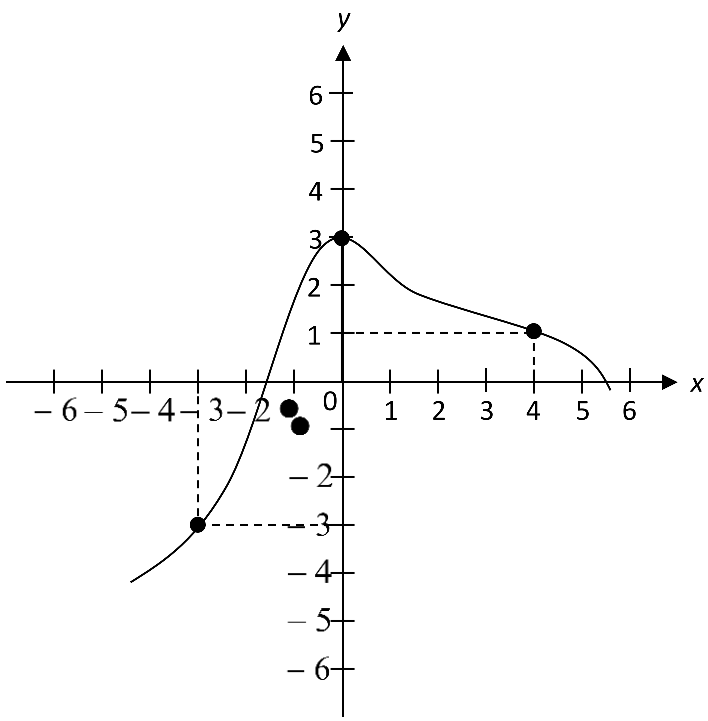


Note: Even though $\sqrt{4} = 2$ or -2, when we write $y = \sqrt{x}$, we refer to only the *+ve* values of *y*. If we want the *–ve* values, then we write $y = -\sqrt{x}$.

2.3 Getting Information from the Graph of a Function

Values of a Function

Suppose the graph below depicts a function $y = f(x)$.



We can write $f(-3) = -3$, $f(0) = 3$, $f(4) = 1$, …

Increasing and Decreasing Functions

A function *f* is said to be **increasing** on an interval *I* if

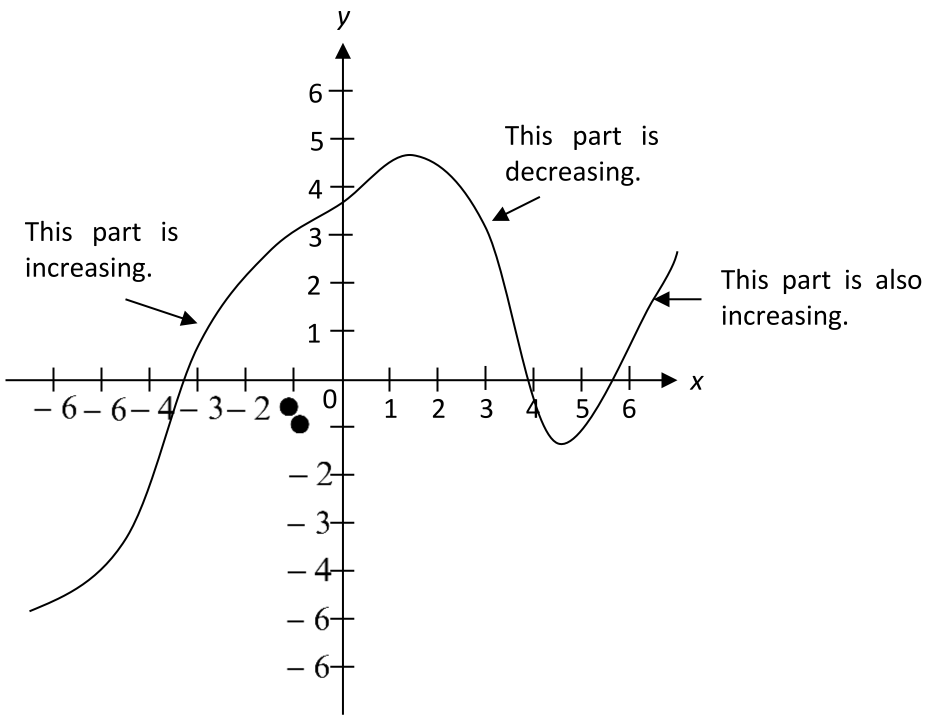
$f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in *I*

It is said to be **decreasing** on *I* if

$f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in *I*

In layman's language:

If the part of the graph goes from bottom left to top right, then that part is increasing.
If the part of the graph goes from top left to bottom right, then that part is decreasing.



Another way of saying it is:

On a particular interval,

1. If *x* increases and *y* also increases, then the function on this interval is increasing.
2. If *x* increases but *y* decreases, then the function on this interval is decreasing.

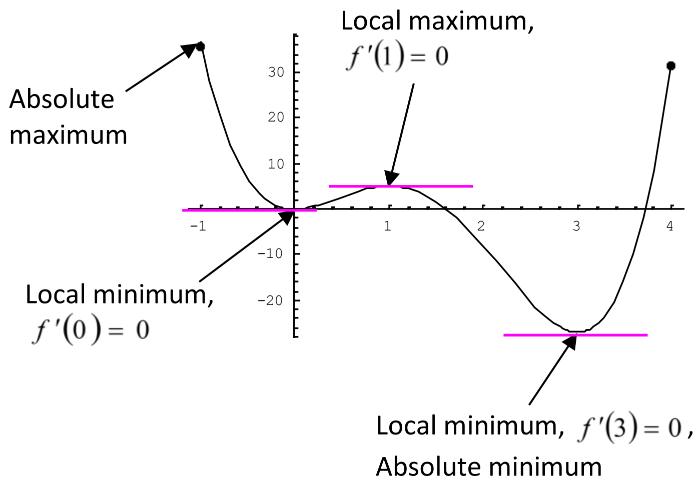
Local Maximum and Minimum Values of a Function

Definition:

A function *f* has a **local** or **relative maximum** at *c* if $f(c) \geq f(x)$ when *x* is near, or within the neighbourhood of, *c*.
Similarly, *f* has a **local** or **relative minimum** at *c* if $f(c) \leq f(x)$ when *x* is near, or within the neighbourhood of, *c*.

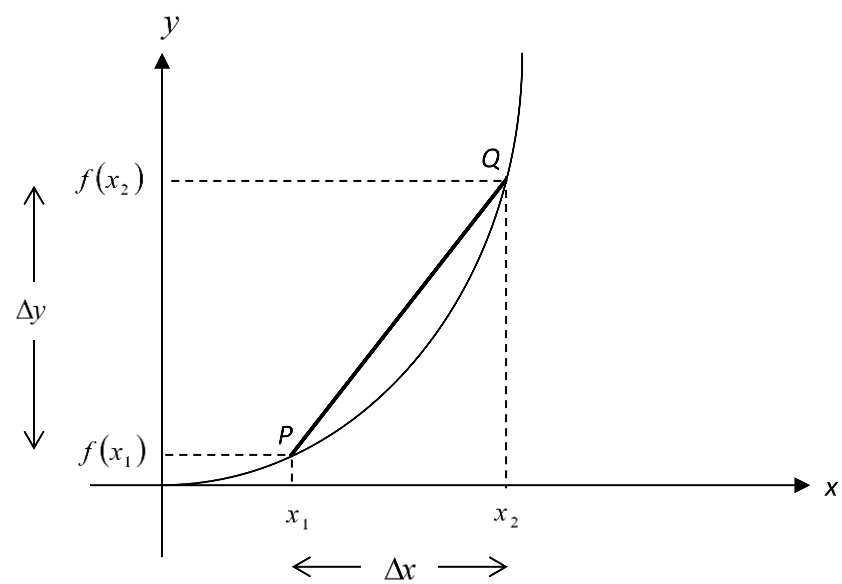
Example in the graph below:

$$y = 3x^4 - 16x^3 + 18x^2,\ x \in [-1,4]$$



2.4 Average Rate of Change of a Function

Let us look at the graph of $y = f(x)$ where point $P(x_1, f(x_1))$ changes to point $Q(x_2, f(x_2))$ when there is a change in the value of *x*.



When *x* changes in value from x_1 to x_2, the change in *x* is

$$\Delta x = x_2 - x_1$$

and the corresponding change in *y* is

$$\Delta y = f(x_2) - f(x_1)$$

The slope of the secant line *PQ* is the quotient

$$m_{PQ} = \frac{Rise}{Run} = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

called the **Average Rate of Change of *y* with respect to *x*** over the interval $[x_1, x_2]$.

Examples:

[1] If you travel 100 miles in two hours, then your average speed for the trip is

$$Average\ Speed = \frac{100\ miles}{2\ hours} = 50\ miles\ per\ hour$$

[2] The average rate of change of the function $f(x) = x^2$ between $x = 2$ and $x = 6$ is

$$Average\ rate\ of\ change = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Where we let $x_1 = 2$ and $x_2 = 6$.

Thus, $Average\ rate\ of\ change = \frac{f(6)-f(2)}{6-2}$

$$= \frac{6^2-2^2}{6-2}$$

$$= \frac{36-4}{4}$$

$$= \frac{32}{4} = 8$$

2.5 Transformations of Functions

Vertical and Horizontal Shifts: Suppose we know what the graph of $y = f(x)$ looks like and suppose we are given $c > 0$. To obtain the graph of

i. $y = f(x) + c$, shift the graph of $y = f(x)$ a distance of *c* units upward
ii. $y = f(x) - c$, shift the graph of $y = f(x)$ a distance of *c* units downward
iii. $y = f(x - c)$, shift the graph of $y = f(x)$ a distance of *c* units to the right
iv. $y = f(x + c)$, shift the graph of $y = f(x)$ a distance of *c* units to the left

Reflecting about the *x*-axis and the *y*-axis: Suppose we know what the graph of $y = f(x)$ looks like. Then, to obtain the graph of

i. $y = -f(x)$, reflect the graph of $y = f(x)$ about the *x*-axis
ii. $y = f(-x)$, reflect the graph of $y = f(x)$ about the *y*-axis

Vertical and Horizontal Stretching and Shrinking: Suppose we know what the graph of $y = f(x)$ looks like and suppose we are given $c > 1$. To obtain the graph of

i. $y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of *c*
ii. $y = \dfrac{1}{c} f(x)$, compress the graph of $y = f(x)$ vertically by a factor of *c*
iii. $y = f(cx)$, compress the graph of $y = f(x)$ horizontally by a factor of *c*
iv. $y = f\left(\dfrac{x}{c}\right)$, stretch the graph of $y = f(x)$ horizontally by a factor of *c*

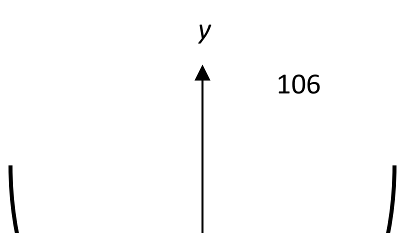
Examples of combining shifting, stretching and reflecting of graphs

[1] Sketch the graph of $y = 1 - 2x - x^2$

Before we perform any transformation from the standard function $y = x^2$, we need to rewrite $y = 1 - 2x - x^2$ into the form $y = a(x - h)^2 + k$, i.e. we need to complete the squares.

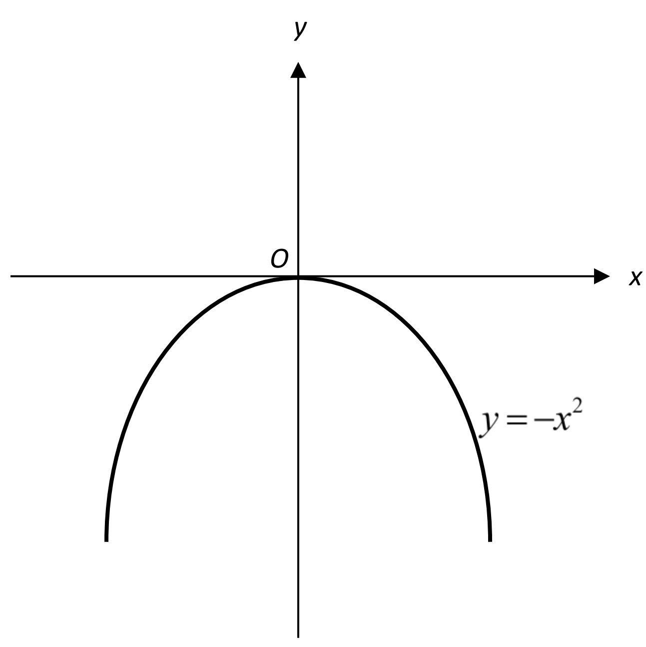
$$\begin{aligned}
y &= 1 - 2x - x^2 \\
&= -x^2 - 2x + 1 \\
&= -(x^2 + 2x) + 1 \\
&= -(x^2 + 2x + 1 - 1) + 1 \\
&= -(x^2 + 2x + 1) + (-1)(-1) + 1 \\
&= -(x^2 + 2x + 1) + 1 + 1 \\
&= -(x^2 + 2x + 1) + 2 \\
&= -(x + 1)^2 + 2
\end{aligned}$$

Now, we can start with $y = x^2$. This is a “smiley” curve with its vertex at the origin $O(0,0)$.

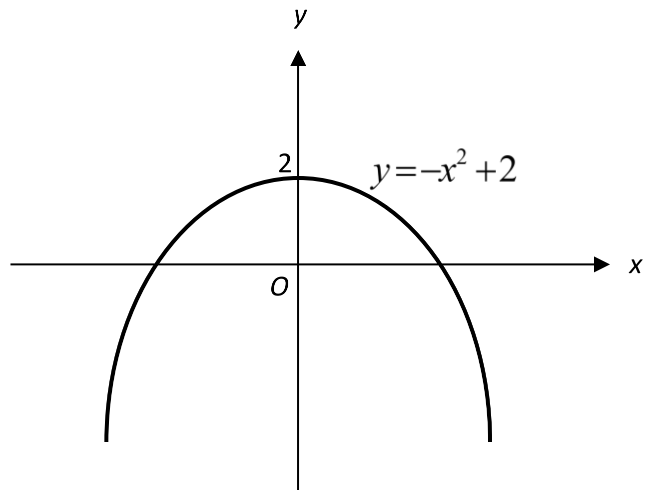


We perform the following transformations:

1. Reflecting of the graph about the *x*-axis. The graph becomes $y = -x^2$.



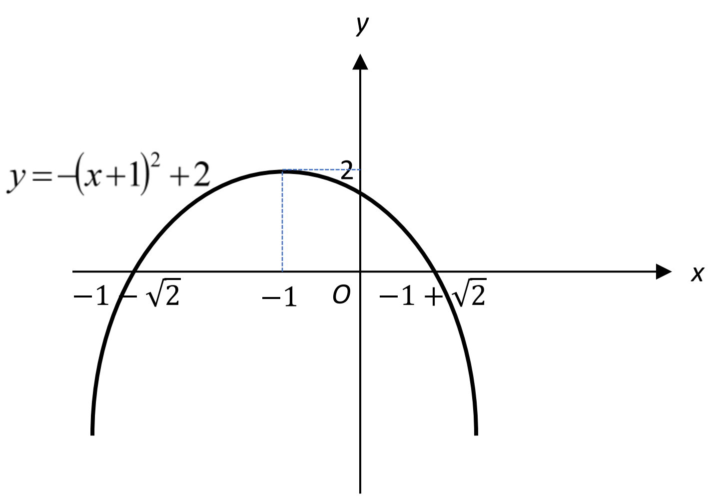
2. Shifting of the graph upward by 2 units. This gives us $y = -x^2 + 2$.



3. Shifting of the previous graph leftwards by 1 unit. This transforms the above graph to that of $y=-(x+1)^2+2$

The *x*-axis intersections are calculated from

$$-(x+1)^2+2=0$$
$$\Rightarrow \quad (x+1)^2=2$$
$$\Rightarrow \quad x+1=\sqrt{2} \text{ or } x+1=-\sqrt{2}$$
$$\Rightarrow \quad x=-1+\sqrt{2} \text{ or } x=-1-\sqrt{2}$$



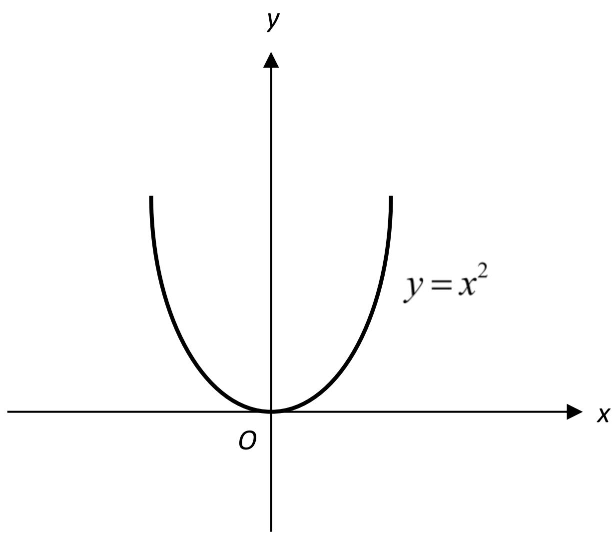
[2] Another example: $y=\left|x^2-2x\right|$

First of all, we sketch the graph of the function without the absolute sign, i.e.

$$y = x^2 - 2x$$

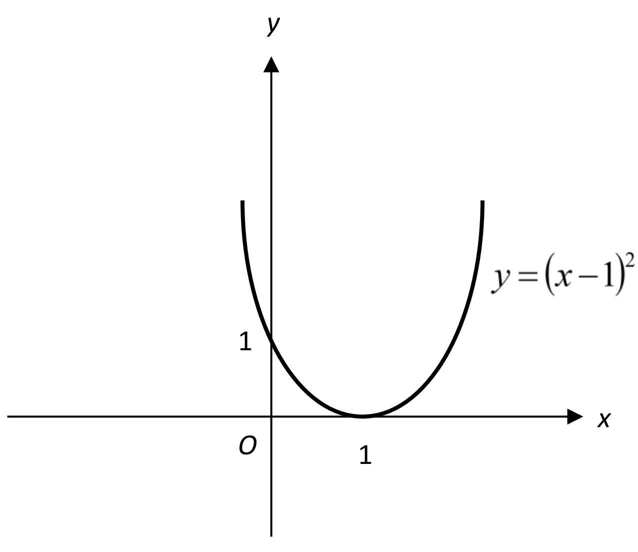
$$\Rightarrow \qquad y = x^2 - 2x + 1 - 1 = (x-1)^2 - 1$$

Since the function involves a quadratic, we start with the graph of $y = x^2$

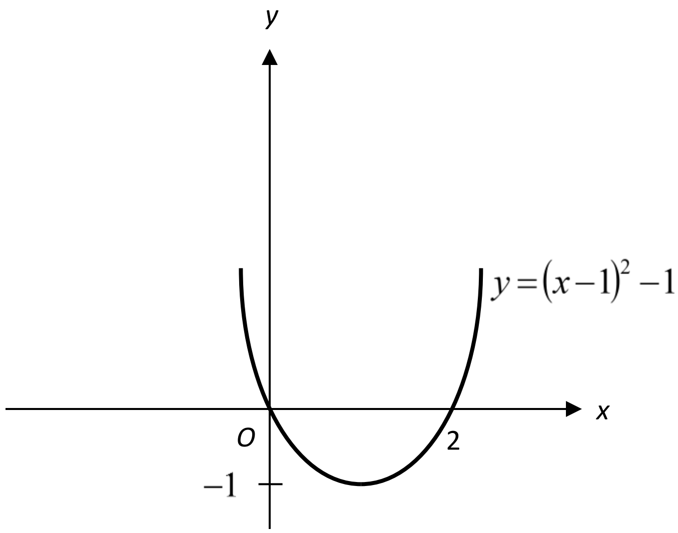


In order to change the graph to that of $y = \left|x^2 - 2x\right|$, we employ the following transformations:

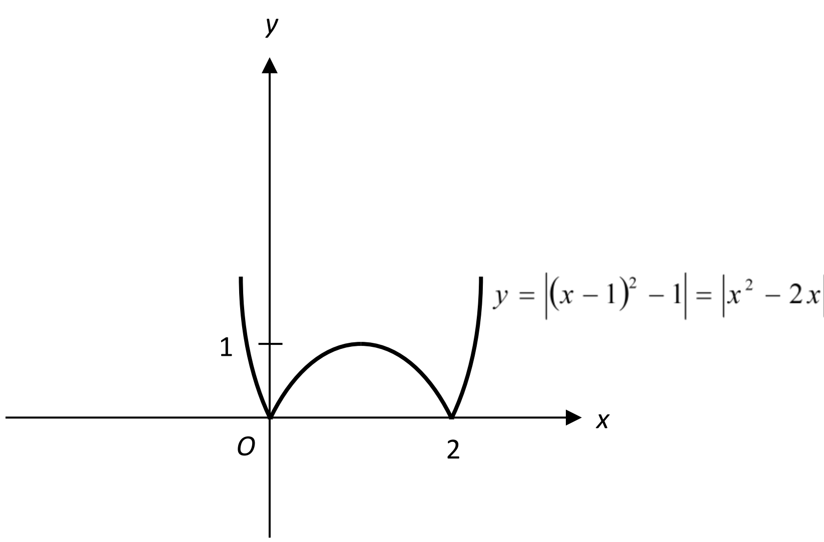
1. We shift the graph of $y = x^2$ one unit to the right. This gives us the graph of $y = (x-1)^2$



2. Then, we shift the graph one unit downward. This gives us the graph of $y = (x-1)^2 - 1 = x^2 - 2x$



3. Finally, we reflect the part of the graph below the *x*-axis, or where *y* is *–ve*, about the *x*-axis. This ensures that whatever is *–ve* in the range becomes *+ve* just like what the absolute function does. Recall that the absolute function changes a negative value into its positive counterpart.



Even and Odd Functions

1. Even function

A function is called an **even** function if it satisfies

$$f(-x) = f(x)$$

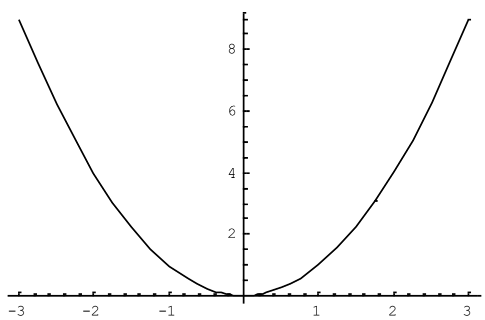
The graph of an even function is symmetric about the *y*-axis.

Example: $f(x) = x^2$

How do we show it is even? We replace *x* with $-x$ in the function.

$$f(-x) = (-x)^2 = x^2 = f(x)$$

$\therefore$ It is an even function because it satisfies the even function test.



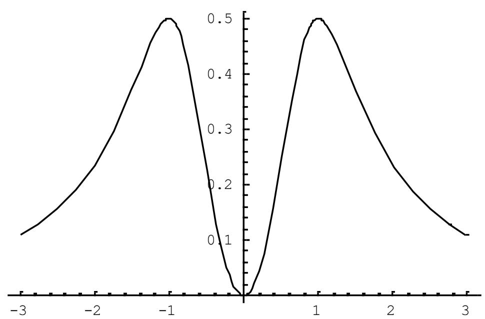
Notice that the left and right sides of the graph are reflections about the *y*-axis.

Another example: $f(x)=\dfrac{x^2}{x^4+1}$

We replace x with $-x$,

$$f(-x)=\frac{(-x)^2}{(-x)^4+1}=\frac{x^2}{x^4+1}=f(x)$$

$\therefore$ This is an even function.



Once again, notice that the graph is symmetric about the *y*-axis.

2. Odd function

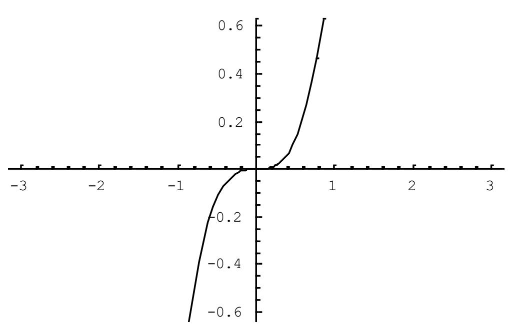
A function is called an **odd** function if it satisfies

$$f(-x)=-f(x)$$

The graph of an odd function is point symmetric about $O(0,0)$.

Example: $f(x)=x^3$

To show it is odd, $f(-x)=(-x)^3=-x^3=-f(x)$

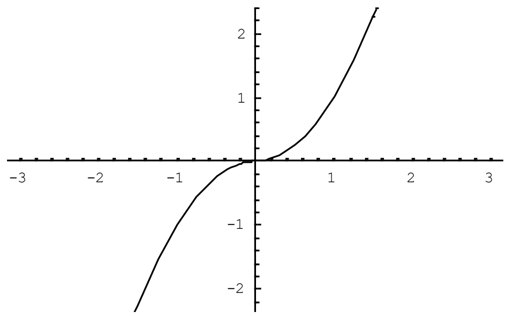


Notice that the graph overlaps onto itself again after a 180° turn about the origin $O(0,0)$.

Another example: $f(x) = x|x|$

$$\Rightarrow \qquad f(-x) = -x|-x| = -x|x| = -f(x)$$

$\therefore$ This is an odd function



Once again, notice that the graph overlaps onto itself again after a 180° turn about $O(0,0)$.

2.6 Combining Functions

Sums, Differences, Products and Quotients

Similar way to adding, subtracting, multiplying and dividing numbers.

1. $(f \pm g)(x) = f(x) \pm g(x)$

2. $(fg)(x) = f(x)g(x)$

3. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$; provided $g(x) \neq 0$

If the domain of $f(x)$ is *A* and the domain of $g(x)$ is *B*,

i. the domain of $f(x) \pm g(x)$ and $(fg)(x)$ is $A \cap B$

ii. the domain of $\left(\frac{f}{g}\right)(x)$ is also $A \cap B$ provided $g(x) \neq 0$

Example:

Given $f(x) = \sqrt{3-x}$ and $g(x) = \sqrt{x^2 - 1}$.

From $f(x) = \sqrt{3-x}$

To find its domain, we know that $3 - x \geq 0$

$$\Rightarrow \quad x \leq 3$$
$$\Rightarrow \quad x \in (-\infty, 3]$$

From $g(x) = \sqrt{x^2 - 1}$

To find its domain, we know that $x^2 - 1 \geq 0$

$$\Rightarrow \quad x^2 \geq 1$$
$$\Rightarrow \quad |x| \geq 1$$
$$\Rightarrow \quad x \geq 1 \text{ or } x \leq -1$$
$$\Rightarrow \quad x \in (-\infty, -1] \cup [1, \infty)$$

i. $(f+g)(x) = f(x) + g(x) = \sqrt{3-x} + \sqrt{x^2 - 1}$

ii. $(f-g)(x) = f(x) - g(x) = \sqrt{3-x} - \sqrt{x^2 - 1}$

iii. $(fg)(x) = f(x)g(x) = \sqrt{3-x} \times \sqrt{x^2-1}$

The domain for (i), (ii) and (iii) is $x \in \{(-\infty, 3]\} \cap \{(-\infty, -1] \cup [1, \infty)\}$

$\Rightarrow \quad x \in (-\infty, -1] \cup [1, 3]$

iv. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{3-x}}{\sqrt{x^2-1}}$

The domain for (iv) is similar to the domain for (i), (ii) and (iii) with the added condition that $g(x) \neq 0$.
Therefore, $x^2 - 1 \neq 0$

$\Rightarrow \quad x \neq -1$ and $x \neq 1$.

The domain for (iv) therefore is $x \in (-\infty, -1) \cup (1, 3]$

Compositions of Functions

Definition

Given two functions *f* and *g*, the **composite** function $f \circ g$, also called the **composition** of *f* and *g*, is defined by

$$(f \circ g)(x) = f(g(x))$$

This means we substitute *x* with $g(x)$ where every *x* occurs in *f*.

The domain of $f \circ g$ is the set of all *x* in the domain of $g(x)$ such that $g(x)$ is in the domain of $f(x)$.

Examples:

Given functions $f(x) = x^2 + 2x - 1$ and $g(x) = 2x + 1$.

1. $(f \circ g)(x) = f(g(x))$

$$= f(2x+1)$$

$$= (2x+1)^2 + 2(2x+1) - 1$$

How does it work? We replace every where *x* occurs in *f* with the new expression of the variable. In this case, we replace *x* with $2x + 1$.

$$f(x) \quad = \quad x^2 \quad + \quad 2x \quad - 1$$

x is replaced with $2x + 1$

$\Rightarrow \quad f(2x+1) = (2x+1)^2 + 2(2x+1) - 1$

Continuing,

$$(f \circ g)(x) = (2x+1)^2 + 2(2x+1) - 1$$

$$= 4x^2 + 4x + 1 + 4x + 2 - 1$$

$$= 4x^2 + 8x + 2$$

2. $(g \circ f)(x) = g\big(f(x)\big)$

$$= g(x^2 + 2x - 1)$$

$$= 2(x^2 + 2x - 1) - 1$$

Here again, we replace every where that *x* occurs in *g* with the new expression of the variable. In this case, we replace *x* with $x^2 + 2x - 1$.

$$g(x) \quad = \quad 2x \quad + 1$$

x is replaced with $x^2 + 2x - 1$

$\Rightarrow$ $g(x^2 + 2x - 1) = 2(x^2 + 2x - 1) + 1$

Continuing,

$$g(x^2 + 2x - 1) = 2(x^2 + 2x - 1) + 1$$

$$= 2x^2 + 4x - 2 + 1$$

$$= 2x^2 + 4x - 1$$

3. $(f \circ f)(x) = f\big(f(x)\big)$

$$= f(x^2 + 2x - 1)$$

$$= (x^2 + 2x - 1)^2 + 2(x^2 + 2x - 1) - 1$$

$$= x^4 + 4x^2 + 1 + 4x^3 - 2x^2 - 4x + 2x^2 + 4x - 2 - 1$$

We use the identity: $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$

Continuing,

$$f(x^2 + 2x - 1) = x^4 + 4x^2 + 1 + 4x^3 - 2x^2 - 4x + 2x^2 + 4x - 2 - 1$$

$$= x^4 + 4x^3 + 4x^2 - 2$$

4. $(g \circ g)(x) = g\big(g(x)\big)$

$$= g(2x+1)$$

$$= 2(2x+1)+1$$

$$= 4x+2+1 = 4x+3$$

Decomposing a Composite Function

This is the opposite of getting a composite function. Suppose we are given a function that is a composite of two functions, i.e. $h(x) = f\big(g(x)\big)$. Our task now is to find what the functions $f(x)$ and $g(x)$ are.

The decomposition of a function is not unique. There can be more than one possibility for $f(x)$ and $g(x)$ given $h(x)$.

The technique to getting the decomposition is to

- Let $g(x)$ take the form of the inner expression
- And $f(x)$ take the form of the outer expression.

Example:

Write the function given by $h(x) = \frac{1}{(x-2)^2}$ as a composition of two functions.

One possible solution:

We can take the inner expression as

$$g(x) = x - 2$$

And the outer expression becomes

$$f(x) = \frac{1}{x^2}$$

Checking: By taking $h(x) = f\big(g(x)\big)$, we have

$$h(x) = f(x-2)$$

$$= \frac{1}{(x-2)^2}$$

Another possible solution:

We take the inner expression as

$$g(x) = (x-2)^2$$

And the outer expression becomes

$$f(x) = \frac{1}{x}$$

Checking: By taking $h(x) = f\big(g(x)\big)$, we have

$$h(x) = f((x-2)^2)$$

$$= \frac{1}{(x-2)^2}$$

Application example

A stone is dropped in a lake, creating a circular ripple that travels outward at a speed of 60 cm/s.

(a) Find a function *g* that models the radius as a function of time.

(b) Find a function *f* that models the area of the circle as a function of the radius.

(c) Find $f \circ g$. What does this function represent?

Solutions:

[a] $g(t) = 60t\ cm$, where *t* is in seconds

[b] $f(r) = \pi r^2$, where *r* is the radius of the circle.

[c] $f \circ g = f\big(g(t)\big) = \pi\big(g(t)\big)^2 = \pi(60t)^2 = 3600\pi t^2$

This is the area of the circle at time *t* seconds.

2.7 One-to-one Functions and Their Inverses

We already know that when we have a function

$$y = f(x)$$

We put a value of *x* into the function and get a value of *y* out of it.

For example:
I go to the shop to buy apples. I want to buy four apples. How much will it cost me?

The opposite can also happen. If we are given the value of *y*, we would like to know which value of *x* will give us this value of *y*.

Opposite of the above example:
I go to the shop to buy apples. I want to buy $5.00 worth of apples. How many apples can I get from that $5.00?

So we need what is known as an inverse function of $f(x)$, which we can write as

$$x = f^{-1}(y)$$

Important Note:

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

One-to-one function

A function *f* is called a **one-to-one function** if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever} \quad x_1 \neq x_2$$

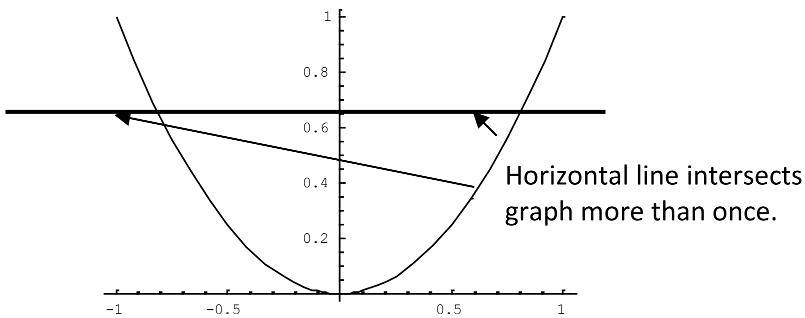
Recall that for a relationship to be defined as a function, for every *x*, there can only be one corresponding *y*. All functions, including one-to-one functions, have this characteristic. They must pass the vertical line test.

One-to-one functions have an extra characteristic, i.e. for every *y*, there can only be one corresponding *x*. One-to-one functions must also pass the horizontal line test.

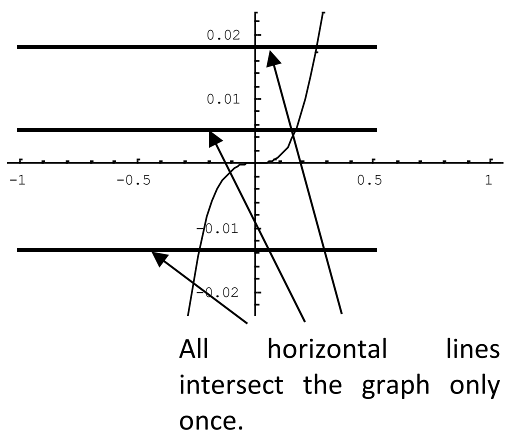
Horizontal Line Test

A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Example: $y = x^2$ is **not** a one-to-one function because there are horizontal lines that intersect the graph more than once.

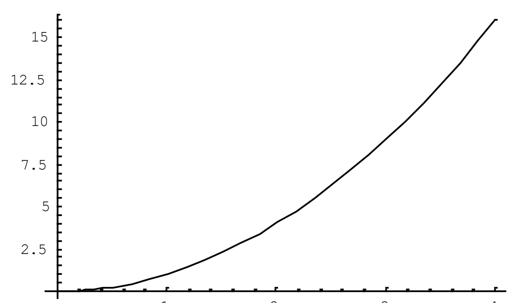


Another example: $y = x^3$ is a one-to-one function because there are no horizontal lines that intersect the graph more than once.

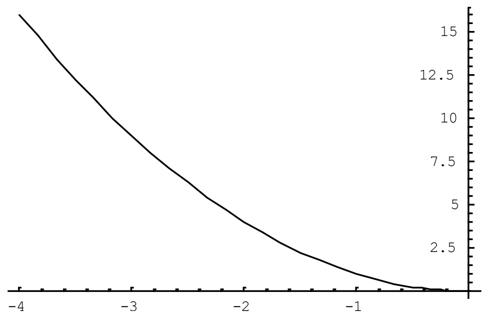


Some functions are not one-to-one on its entire domain. But it may be a one-to-one function on part of its domain.

Example: $y = x^2$ is one-to-one on $x \geq 0$ or $x \in [0, \infty)$



and $y = x^2$ is also one-to-one on $x \le 0$ or $x \in (-\infty, 0]$



By definition:

Only one-to-one functions possess inverse functions.

Domain and Range of an inverse function

Let *f* be a one-to-one function with domain *A* and range *B*. Then its **inverse function** f^{-1} has domain *B* and range *A* and is defined by

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$

for any *y* in *B*.

What the above says is:

- The domain of a one-to-one function becomes the range of its inverse function.
- The range of a one-to-one function becomes the domain of its inverse function.

We usually write the inverse function as

$$f^{-1}(x) = y$$

because we are used to using *x* as the symbol of the independent variable and *y* as the symbol of the dependent variable.

What does an inverse function do?

$$f(1) = 5 \quad \Leftrightarrow \quad f^{-1}(5) = 1$$

$$f(3) = 7 \quad \Leftrightarrow \quad f^{-1}(7) = 3$$

$$f(8) = -10 \quad \Leftrightarrow \quad f^{-1}(-10) = 8$$

Common sense (I hope) and cancellation equations:

For every $f(x)$ that is a one-to-one function with domain *A* and range *B*, we have

$$f^{-1}(f(x)) = x \text{ , for } x \in A$$

and $$f(f^{-1}(x)) = x \text{ , for } x \in B$$

How to find the inverse function of a one-to-one function $f(x)$

1. Write $y = f(x)$
2. Rewrite the equation as *x* in terms of *y* (if possible)
3. Interchange *x* and *y*. This gives us $y = f^{-1}(x)$

Examples:

Find a formula for the inverse of the function.

[1] $f(x) = \dfrac{4x-1}{2x+3}$

i. Write $y = f(x)$

$$y = \frac{4x-1}{2x+3}$$

ii. Rewrite the equation as *x* in terms of *y*.

From, $y = \dfrac{4x-1}{2x+3}$

$\Rightarrow \quad y(2x+3) = 4x-1$

$\Rightarrow \quad 2xy + 3y = 4x - 1$

$\Rightarrow \quad 4x - 2xy = 3y + 1$

$\Rightarrow \quad 2x(2-y) = 3y + 1$

$\Rightarrow \quad x = \dfrac{3y+1}{2(2-y)}$

iii. Interchange *x* and *y*.

From step ii, after interchanging *x* and *y*, we have

$$y = \frac{3x+1}{2(2-x)}$$

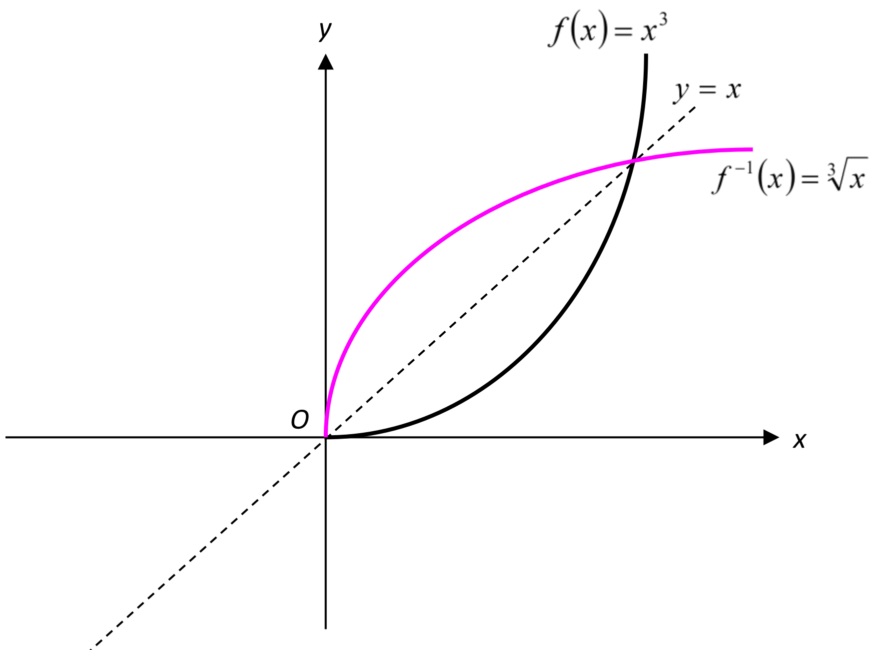
$$\Rightarrow \quad f^{-1}(x) = \frac{3x+1}{2(2-x)}$$

Graph of the inverse function

The graph of $f^{-1}(x)$ is obtained by reflecting the graph of $f(x)$ about the line $y = x$.

Example:

The inverse of $f(x) = x^3$ is $f^{-1}(x) = \sqrt[3]{x}$.



3.1 Quadratic Functions and Models

Quadratic functions are polynomials of degree two.

$$f(x) = ax^2 + bx + c \text{ where } a \neq 0$$

is a quadratic function.

Graphing Quadratic Functions Using the Standard Form

Properties of quadratic functions

1. All quadratic functions $f(x) = ax^2 + bx + c$

   can be rewritten as $f(x) = a(x-h)^2 + k$ (The Standard Form of a Quadratic Function)

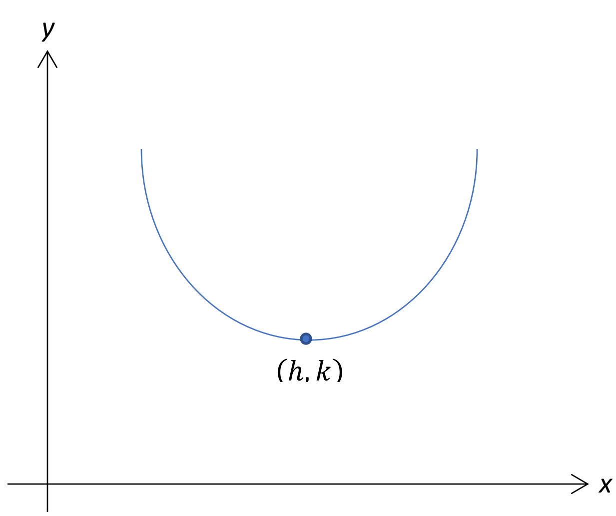
   by performing the ***Completing The Square*** process.

   The vertex of the parabola is always (h, k).

2. All quadratic functions yield parabolic graphs. There are two possibilities

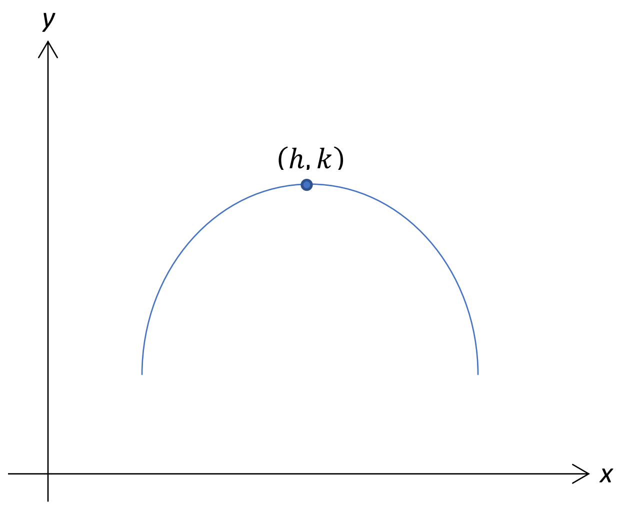
   i. Concave upward graphs

      These occur when $a > 0$.



ii. Concave downward graphs

These occur when $a < 0$.



Completing the Square

Examples:

Rewrite the following quadratic functions in standard form:

1. $f(x) = ax^2 + bx + c$

Step #1: We factor out the coefficient of x^2, i.e. *a*, from the first two terms.

$$f(x) = ax^2 + bx + c$$

$$= a\left(x^2 + \frac{b}{a}x\right) + c$$

Step #2: Take the coefficient of *x* within the brackets, i.e. $\frac{b}{a}$, and halve it.

$$\frac{b}{2a}$$

Step #3: Square the result from Step #2, i.e. we square $\frac{b}{2a}$.

$$\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

Step #4: Add and subtract the result from Step #3, i.e. $\left(\frac{b}{2a}\right)^2$, within the brackets.

$$f(x) = a\left(x^2 + \frac{b}{a}x\right) + c$$

$$= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c$$

Step #5: Take the last term out of the expression within the brackets. Or we can think of this step as distributing *a* into $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$ and into $-\frac{b^2}{4a^2}$.

$$f(x) = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + (a)\left(-\frac{b^2}{4a^2}\right) + c$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a} + c$$

Step #6: The expression within the brackets is a perfect square.

$$f(x) = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a} + c$$

$$= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

$\therefore$ $$f(x) = ax^2 + bx + c$$

$$= a\left(x - \left(-\frac{b}{2a}\right)\right)^2 + c - \frac{b^2}{4a}$$

The vertex for this quadratic's parabola is at the point $(h, k) = \left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$.

Notice that when $x = -\frac{b}{2a}$,

$$f\left(-\frac{b}{2a}\right) = a\left(-\frac{b}{2a} + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

$$= a(0)^2 + c - \frac{b^2}{4a} = c - \frac{b^2}{4a}$$

$\Rightarrow$ $$f\left(-\frac{b}{2a}\right) = c - \frac{b^2}{4a}$$

$\therefore$ The vertex for this quadratic's parabola can also be calculated as $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

Or we can also rewrite $k = c - \frac{b^2}{4a}$

$$= c - a\left(\frac{b^2}{4a^2}\right)$$

$$= c - a\left(-\frac{b}{2a}\right)^2$$

$$= c - ah^2$$

In summary, we have the vertex of the parabola (h, k) where

$$h = -\frac{b}{2a}$$

and

$$k = c - \frac{b^2}{4a}$$

or

$$k = f\left(-\frac{b}{2a}\right)$$

or

$$k = c - ah^2$$

2. $f(x) = x^2 + 3x + \frac{1}{4}$

Step #1: We factor out the coefficient of x^2, i.e. 1, from the first two terms.

$$f(x) = x^2 + 3x + \frac{1}{4}$$

$$= (x^2 + 3x) + \frac{1}{4}$$

Step #2: Take the coefficient of *x* within the brackets, i.e. 3, and halve it.

$$\frac{3}{2}$$

Step #3: Square the result from Step #2, i.e. we square $\frac{3}{2}$.

$$\left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

Step #4: Add and subtract the result from Step #3, i.e. $\left(\frac{3}{2}\right)^2$, within the brackets.

$$f(x) = \left(x^2 + 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2\right) + \frac{1}{4}$$

$$= \left(x^2 + 3x + \tfrac{9}{4} - \tfrac{9}{4}\right) + \tfrac{1}{4}$$

Step #5: Take the last term out of the expression within the brackets.

$$f(x) = \left(x^2 + 3x + \tfrac{9}{4}\right) + \left(-\tfrac{9}{4}\right) + \tfrac{1}{4}$$

$$= \left(x^2 + 3x + \tfrac{9}{4}\right) - \tfrac{8}{4}$$

$$= \left(x^2 + 3x + \tfrac{9}{4}\right) - 2$$

Step #6: The expression within the brackets is a perfect square.

$$f(x) = \left(x^2 + 3x + \tfrac{9}{4}\right) - 2$$

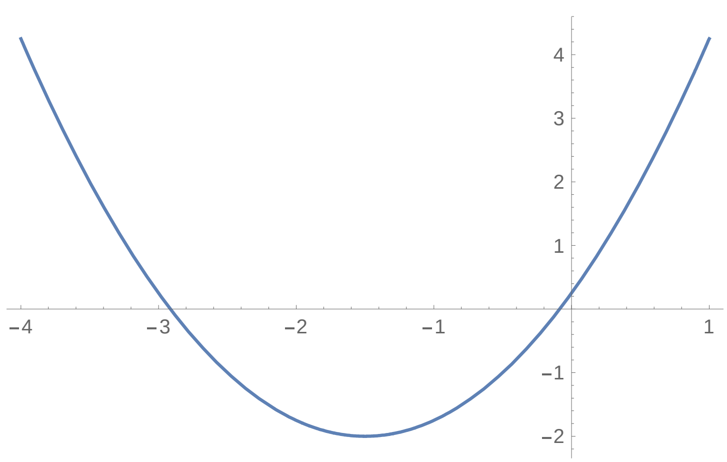
$$= \left(x + \tfrac{3}{2}\right)^2 - 2$$

$$\therefore \qquad f(x) = x^2 + 3x + \tfrac{1}{4}$$

$$= \left(x + \tfrac{3}{2}\right)^2 - 2$$

$$f(x) = \left(x - \left(-\tfrac{3}{2}\right)\right)^2 - 2$$

The graph of this quadratic's parabola is concave up, since $a = 1$ which is positive, and this parabola has a vertex at $\left(-\frac{3}{2}, -2\right)$.



3. $f(x) = -x^2 + 2x + 5$

Step #1: We factor out the coefficient of x^2, i.e. -1, from the first two terms.

$$f(x) = -x^2 + 2x + 5$$

$$= -(x^2 - 2x) + 5$$

Step #2: Take the coefficient of *x* within the brackets, i.e. -2, and halve it.

$$\frac{-2}{2} = -1$$

Step #3: Square the result from Step #2, i.e. we square -1 .

$$(-1)^2 = 1$$

Step #4: Add and subtract the result from Step #3, i.e. $(-1)^2$, within the brackets.

$$f(x) = -(x^2 - 2x + (-1)^2 - (-1)^2) + 5$$

$$= -(x^2 - 2x + 1 - 1) + 5$$

Step #5: Take the last term out of the expression within the brackets.

$$f(x) = -(x^2 - 2x + 1) - (-1) + 5$$

$$= -(x^2 - 2x + 1) + 1 + 5$$

$$= -(x^2 - 2x + 1) + 6$$

Step #6: The expression within the brackets is a perfect square.

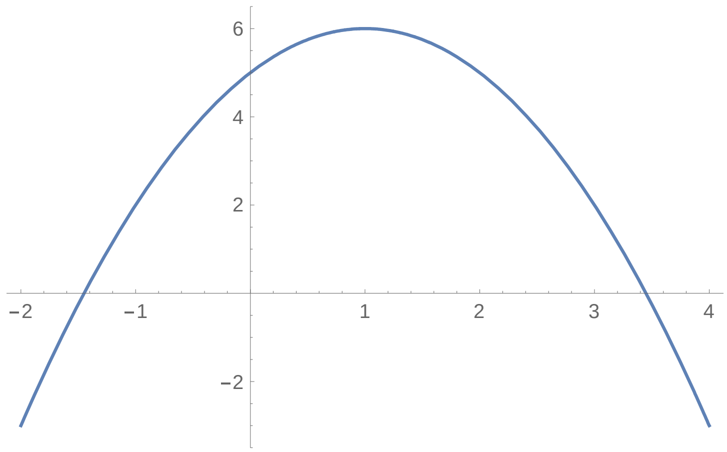
$$f(x) = -(x^2 - 2x + 1) + 6$$

$$= -(x - 1)^2 + 6$$

$\therefore$ $$f(x) = -x^2 + 2x + 5$$

$$= -(x - 1)^2 + 6$$

The graph of this quadratic's parabola is concave down, since $a = -1$ which is negative, and this parabola has a vertex at (1,6).



4. $f(x) = 2x^2 - x + 1$

Step #1: We factor out the coefficient of x^2, i.e. 2, from the first two terms.

$$f(x) = 2x^2 - x + 1$$

$$= 2\left(x^2 - \frac{1}{2}x\right) + 1$$

Step #2: Take the coefficient of *x* within the brackets, i.e. $-\frac{1}{2}$, and halve it.

$$\frac{-\frac{1}{2}}{2} = -\frac{1}{4}$$

Step #3: Square the result from Step #2, i.e. -1 .

$$\left(-\frac{1}{4}\right)^2 = \frac{1}{16}$$

Step #4: Add and subtract the result from Step #3, i.e. $\left(-\frac{1}{4}\right)^2$, within the brackets.

$$f(x) = 2\left(x^2 - \frac{1}{2}x + \left(-\frac{1}{4}\right)^2 - \left(-\frac{1}{4}\right)^2\right) + 1$$

$$= 2\left(x^2 - \frac{1}{2}x + \frac{1}{16} - \frac{1}{16}\right) + 1$$

Step #5: Take the last term out of the expression within the brackets.

$$f(x) = 2\left(x^2 - \frac{1}{2}x + \frac{1}{16} - \frac{1}{16}\right) + 1$$

$$= 2\left(x^2 - \frac{1}{2}x + \frac{1}{16}\right) + 2\left(-\frac{1}{16}\right) + 1$$

$$= 2\left(x^2 - \frac{1}{2}x + \frac{1}{16}\right) - \frac{1}{8} + 1$$

$$= 2\left(x^2 - \frac{1}{2}x + \frac{1}{16}\right) - \frac{1}{8} + \frac{8}{8}$$

$$= 2\left(x^2 - \frac{1}{2}x + \frac{1}{16}\right) + \frac{7}{8}$$

Step #6: The expression within the brackets is a perfect square.

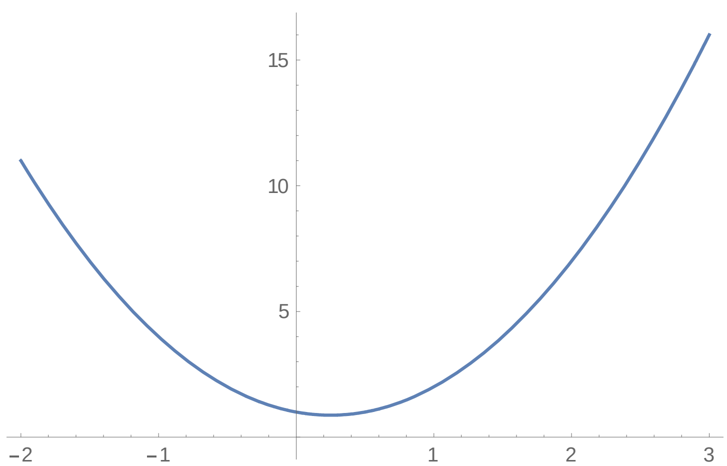
$$f(x) = 2\left(x^2 - \frac{1}{2}x + \frac{1}{16}\right) + \frac{7}{8}$$

$$= 2\left(x - \frac{1}{4}\right)^2 + \frac{7}{8}$$

$\therefore \qquad f(x) = 2x^2 - x + 1$

$$= 2\left(x - \frac{1}{4}\right)^2 + \frac{7}{8}$$

The graph of this quadratic's parabola is concave up, since $a = 2$ which is positive, and this parabola has a vertex at $\left(\frac{1}{4}, \frac{7}{8}\right)$.



Modelling with Quadratic Functions

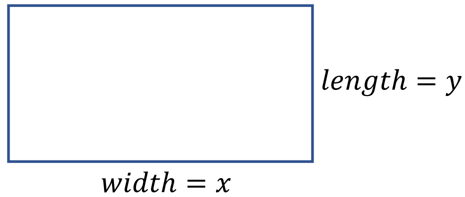
Example

[1] **Fencing a Dog Run** I have 2400 *ft* of fencing to contain a rectangular dog run.

(a) Find a function that models the area of the dog run in terms of the width *x* of the dog run.

(b) Find the dimensions of the rectangle that maximise the area of the dog run.

Solution: Let x be the width and y be the length of the dog run,



(a) Let A be the area of the corral that we want to maximise,

$$A = xy \qquad \text{... (1)}$$

I have 2400 *ft* of fencing, therefore the perimeter of the dog run must be

$$x + y + x + y = 2400$$

$$\Rightarrow \qquad 2x + 2x = 2400$$

$$\Rightarrow \qquad 2(x + y) = 2400$$

$$\Rightarrow \qquad x + y = \frac{2400}{2} = 1200$$

$$\Rightarrow \qquad y = 1200 - x \qquad \text{... (2)}$$

Substituting (2) into (1),

$$A = xy$$

$$\Rightarrow \qquad A = x(1200 - x)$$

$$= -x^2 + 1200x$$

This is a quadratic function with $a = -1$, $b = 1200$ and $c = 0$.

Since $a < 0$, meaning it draws a concave down parabola. Its vertex, (h, k), gives us the maximum point.

To find the vertex, $\quad (h, k) = \left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$

Thus, $\quad h = -\frac{b}{2a} = -\frac{1200}{2(-1)} = 600$

And $\quad k = c - \frac{b^2}{4a} = 0 - \frac{1200^2}{4(-1)} = \frac{1200^2}{2^2} = \left(\frac{1200}{2}\right)^2 = 600^2 = 360000$

∴ The dimensions that maximise the area of the dog run are

$$Width, x = h = 600\ ft$$

And $Length, y = 1200 - x = 1200 - 600 = 600\ ft$, from (2).

The maximum area is $k = 360000\ ft^2$

3.2 Polynomial Functions and Their Graphs

Polynomial Functions

A polynomial function of degree *n* has the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$$

Where	n	is a non-negative integer,
	$a_n x^n$	is the leading term,
	$a_n \neq 0$	is the leading coefficient,
And	a_0	is the constant term.

Examples:

$f(x) = 2x^3 - \frac{1}{2}x^2 + 0.03x - 5$ is a polynomial of degree 3.

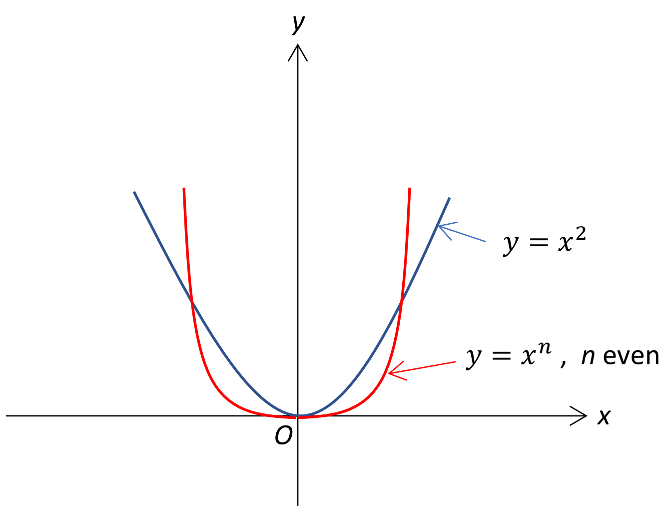
$f(x) = 2x^2 + \frac{2}{x} = 1$ is not a polynomial.

$f(x) = 3x^3 + 2\sqrt{x} - 3$ is not a polynomial.

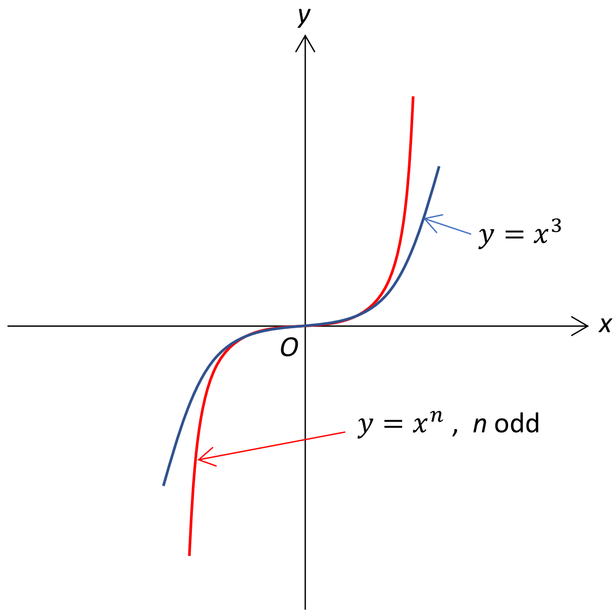
Graphing Basic Polynomial Functions

Graphs of $y = x^n$

1. When *n* is even, the graph of $y = x^n$ looks almost similar to the graph of $y = x^2$.



2. When *n* is odd, the graph of $y = x^n$ looks almost similar to the graph of $y = x^3$



Note: In both cases, the graphs of $y = x^n$ are flatter at the origin $O(0,0)$. The greater *n* is, the flatter the graph.

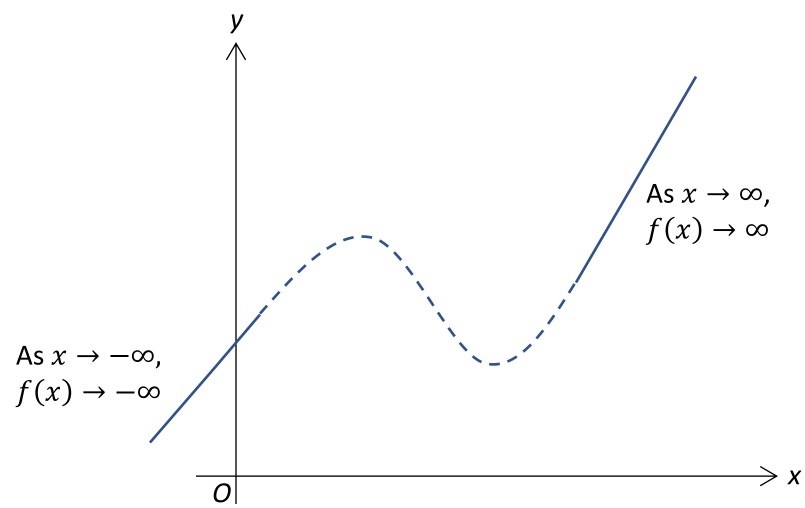
End Behaviour and the Leading Term

The orientation of the two ends of the graph of a polynomial

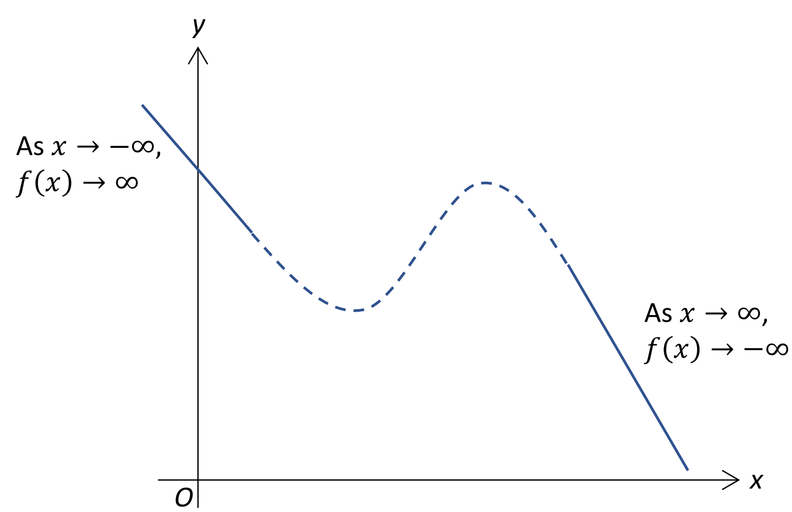
$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

depends on 4 conditions:

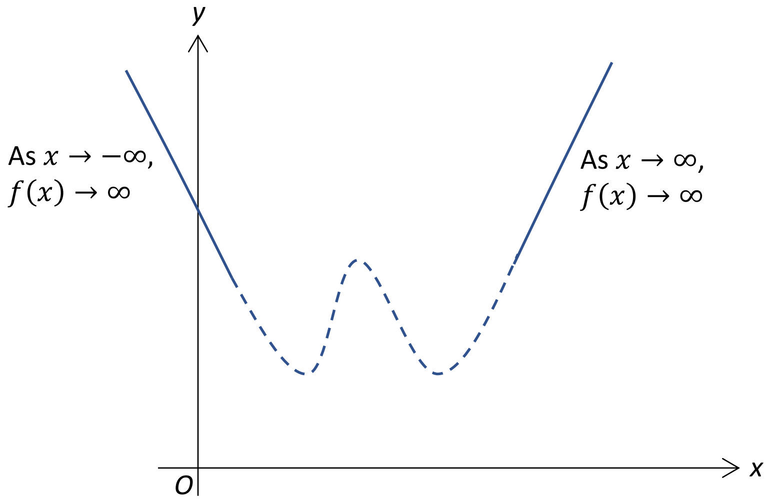
1. When *n* is odd and $a_n > 0$:



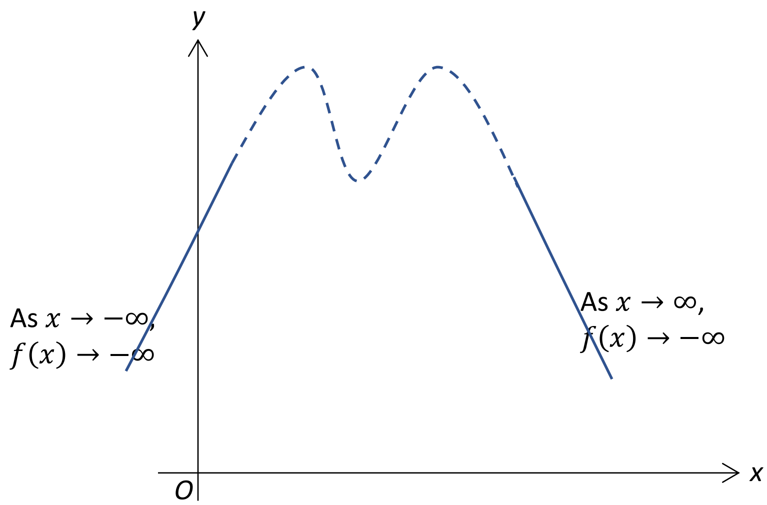
2. When *n* is odd and $a_n < 0$



3. When *n* is even and $a_n > 0$



4. When *n* is even and $a_n < 0$



Notation:
If a polynomial $P(x)$ is such that $P(c) = 0$, then we can say that
[1] *c* is a zero of *P*
[2] $x = c$ is a solution or root of $P(x) = 0$
[3] $x - c$ is a factor of $P(x)$
[4] *c* is an *x*-intercept of the graph of *P*.

Zeroes of Polynomial Functions

A polynomial of degree *n*

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

has the following properties:

1. All polynomials are continuous functions on $(-\infty, \infty)$, i.e. the graphs of all polynomials can be drawn without lifting pen from paper.

2. All polynomials have graphs that are smooth.

3. A polynomial of degree *n* has at the most *n* real zeroes.

4. **Local Extrema of Polynomials:** The graph of a polynomial of degree *n* has at the most $n-1$ number of turning points, or local (relative) maximum or local (relative) minimum points.

Suppose $x = a$ is a zero of the polynomial. This means that this polynomial has a factor

$$(x-a)^k\text{, where } k > 0$$

Two possibilities happen here:

1. If *k* is odd, then the graph of the polynomial crosses the *x*-axis at $x = a$.

2. If *k* is even, then the graph of the polynomial touches the *x*-axis but does not cross the *x*-axis at $x = a$.

Note: *k* is known as the multiplicity of the factor.

We will use these information to sketch the graph of a polynomial.

Example:

Sketch the graph of $y = 2(x-1)(x+1)^2$.

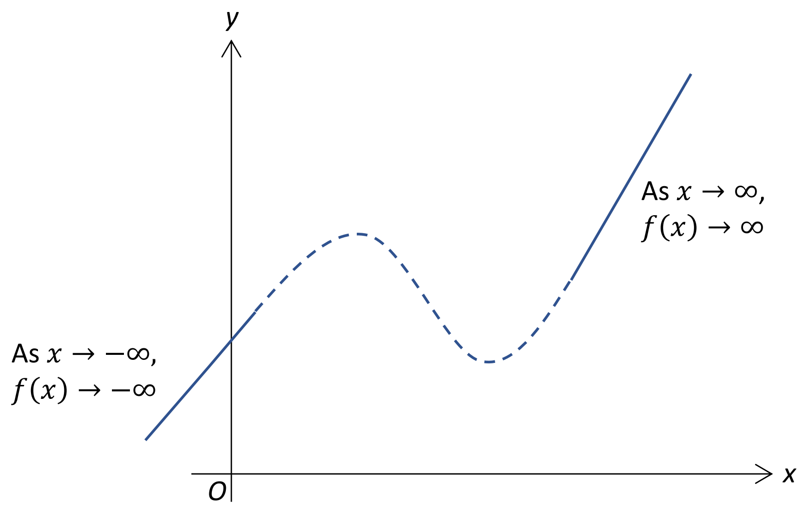
1. **The Leading Coefficient Test.**

   By expanding the polynomial, we obtain

   $$y = 2(x-1)(x+1)^2$$
   $$= 2x^3 + 2x^2 - 2x - 2$$

   We can see that the polynomial is of odd degree, $n = 3$
   and the leading coefficient is positive, i.e. $a_n = 2 > 0$

The two ends of the graph will have the following orientation:



2. **Zeroes of the Polynomial**

Let $\quad y = 2(x-1)(x+1)^2 = 0$

$\Rightarrow \quad x - 1 = 0$ or $x + 1 = 0$

$\Rightarrow \quad x = 1$ or $x = -1$ are the zeroes of the polynomial.

The zero $x = 1$ is of odd multiplicity because the exponent of $x - 1$ is odd.

$\Rightarrow \quad$ The graph crosses the *x*-axis at $x = 1$

The zero $x = -1$ is of even multiplicity because the exponent of $x + 1$ is even.

$\Rightarrow \quad$ The graph touches but does not cross the *x*-axis at $x = -1$

3. Sign table of the polynomial function

We divide the number line into intervals based on the zeroes of the polynomial, i.e. $(-\infty, -1)$, $(-1,1)$ and $(1, \infty)$. Then, on each interval we determine where does the graph fall. Does the graph fall below or above the *x*-axis?

We can easily determine this for each interval by taking any point of *x* within that interval as a test point.

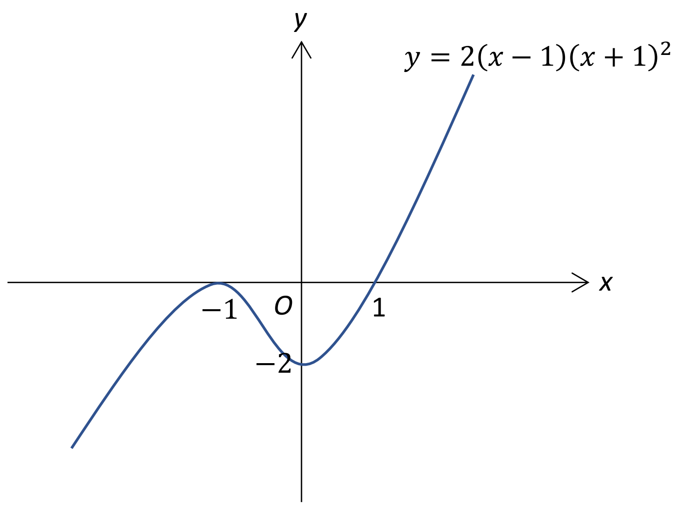
Interval	$(-\infty, -1)$	$(-1,1)$	$(1, \infty)$
Test value	$x = -2$	$x = 0$	$x = 2$
Function value at test point	$f(-2) = -6$	$f(0) = -2$	$f(2) = 18$
Sign of $f(x)$	$-ve$	$-ve$	$+ve$

4. **Determine the *y*-intercept**

When $x = 0$, we have $f(0) = -2$.

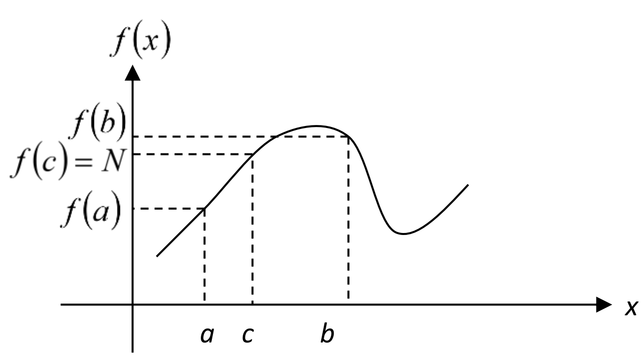
The graph intersects the *y*-axis at $(0, -2)$

5. **Sketch the graph**



The Immediate Value Theorem

Let *a* and *b* be real numbers such that $a < b$. If *f* is a continuous function on the interval $[a, b]$ such that $f(a) \neq f(b)$, then *f* takes on every value between $f(a)$ and $f(b)$ on the interval $[a, b]$.



Application of the Intermediate Value Theorem:

If we are given a continuous function $f(x)$, and $f(a)$ and $f(b)$ have opposite signs,
i.e. either $f(a) = +ve$ and $f(b) = -ve$,
or $f(a) = -ve$ and $f(b) = +ve$,
then there must exist at least one value *c* where $a < c < b$ such that $f(c) = 0$.

Example:

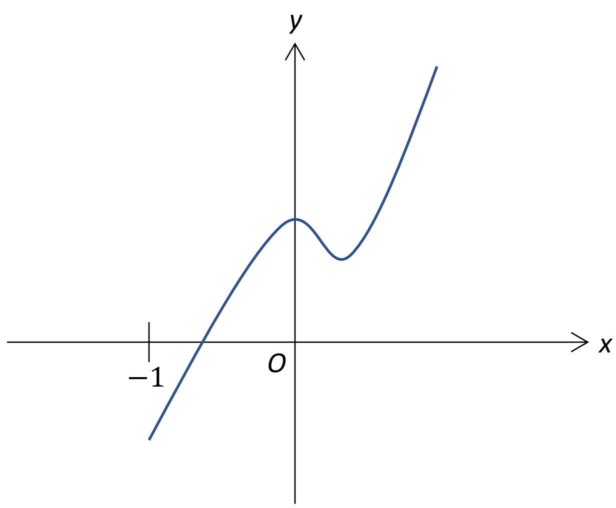
Show that the function $f(x) = x^3 - x^2 + 1$ has at least one zero between $x = -1$ and $x = 0$.

Solution: $f(-1) = (-1)^3 - (-1)^2 + 1 = -1 - 1 + 1 = -1$

$f(0) = 0^3 - 0^2 + 1 = 1$

We can see that $f(-1) < 0 < f(0)$. Since $f(x)$ is a polynomial and therefore continuous, we can conclude from the Intermediate Value Theorem that there must be at least one point between $x = -1$ and $x = 0$ where the value of the function is zero.

Graph of $f(x) = x^3 - x^2 + 1$:



3.3 Dividing Polynomials

A rational function is defined as

$$f(x) = \frac{N(x)}{D(x)}$$

Where both $N(x)$ and $D(x)$ are polynomials,

and $D(x) \neq 0$ on the domain of $f(x)$.

We define proper rationals and improper rationals as follows:

- If $deg\big(N(x)\big) < deg\big(D(x)\big)$, then this rational is **proper**.
- If $deg\big(N(x)\big) \geq deg\big(D(x)\big)$, then this rational is **improper**.

Examples:

1. $\frac{2x+3}{x^2+3x+4}$ is a proper rational because the degree of the numerator is less than the degree of the denominator.

2. $\frac{2x^2+1}{x^2+3x+4}$ is an improper rational because the degree of the numerator is equal to the degree of the denominator.

3. $\frac{2x^3+3}{x^2+3x+4}$ is also an improper rational because the degree of the numerator is greater than the degree of the denominator.

Given an improper rational, we can obtain a proper rational term from it by performing a division. If $\frac{p(x)}{q(x)}$ is an improper rational, we can perform a division to rewrite it as

$$\frac{p(x)}{q(x)} = s(x) + \frac{t(x)}{q(x)}$$

Where $p(x)$ is the Dividend,

$q(x)$ is the Divisor,

$s(x)$ is the Quotient,

$t(x)$ is the Remainder

And $\frac{t(x)}{q(x)}$ is a proper rational expression.

Long Division of Polynomials

Examples:

1. $\frac{x^4-3x^2-1}{x^2+5} = \frac{x^4+0x^3-3x^2+0x-1}{x^2+0x+5}$

Since the divisor is not a linear factor, the safest way to perform this division is by using the long division method.

$$
\begin{array}{ccccc|ccccccccc}
 & & & & & x^2 & + & 0x & - & 8 & & & & \\
\cline{6-14}
x^2 & + & 0x & + & 5 & x^4 & + & 0x^3 & - & 3x^2 & + & 0x & - & 1 \\
 & & & & & x^4 & + & 0x^3 & + & 5x^2 & & & & \\
\cline{6-14}
 & & & & & & & & - & 8x^2 & + & 0x & - & 1 \\
 & & & & & & & & - & 8x^2 & + & 0x & - & 40 \\
\cline{6-14}
 & & & & & & & & & & & & + & 39
\end{array}
$$

$\therefore \quad \frac{x^4-3x^2-1}{x^2+5} = x^2 - 8 + \frac{39}{x^2+5}$

2. $\frac{3x^2+1}{x^2+x+9} = \frac{3x^2+0x+1}{x^2+x+9}$

$$
\begin{array}{ccccc|ccccc}
 & & & & & 3 & & & & \\
\cline{6-10}
x^2 & + & x & + & 9 & 3x^2 & + & 0x & + & 1 \\
 & & & & & 3x^2 & + & 3x & + & 27 \\
\cline{6-10}
 & & & & & & - & 3x & - & 26
\end{array}
$$

$\therefore \quad \frac{3x^2+1}{x^2+x+9} = 3 + \frac{-3x-26}{x^2+x+9}$

Synthetic Division

If the divisor is linear, i.e. $x - c$, we can safely use a synthetic division to divide polynomials, $\frac{P(x)}{x-c}$.

Remainder and Factor Theorems

Remainder Theorem:
If a polynomial $P(x)$ is divided by $x - c$, then the remainder is $P(c)$

Factor Theorem:
If c is a zero of P, i.e. $P(c) = 0$, then $x - c$ is a factor of $P(x)$.

Examples:

3. $\frac{x^4-x^3+x^2-3x-6}{x-1}$ where $P(x) = x^4 - x^3 + x^2 - 3x - 6$ is divided by $x - 1$.

$$\begin{array}{c|ccccc} 1 & 1 & -1 & 1 & -3 & -6 \\ & & 1 & 0 & 1 & -2 \\ \hline & 1 & 0 & 1 & -2 & -8 \end{array}$$

The remainder tells us that $P(1) = -8$

$\therefore \quad \frac{x^4-x^3+x^2-3x-6}{x-1} = x^3 + 0x^2 + x - 2 + \frac{-8}{x-1}$

$= x^3 + x - 2 - \frac{8}{x-1}$

$\Rightarrow \quad x^4 - x^3 + x^2 - 3x - 6 = (x^3 + x - 2)(x - 1) - 8$

4. $\frac{x^3-2x^2+3x-6}{x-2}$

$$\begin{array}{c|cccc} 2 & 1 & -2 & 3 & -6 \\ & & 2 & 0 & 6 \\ \hline & 1 & 0 & 3 & 0 \end{array}$$

Remainder $= 0$

$\therefore \qquad \frac{x^3-2x^2+3x-6}{x-2} = x^2 + 0x + 3 + \frac{0}{x-2}$

$= x^2 + 3$

$\Rightarrow \qquad x^3 - 2x^2 + 3x - 6 = (x^2 + 3)(x - 2)$

3.4 Real Zeroes of Polynomials

Rational Zeroes Theorem

Given the polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

The possible real zeroes are

$$\frac{Factors\ of\ constant\ term,\ a_0}{Factors\ of\ leading\ coefficient, a_n}$$

Let us say we have a value $x = k$ as a possible real zero. To determine whether $x = k$ is indeed a real zero of the polynomial or not, we can perform either one of the following two tasks:

- Assign $x = k$ to the polynomial. If the result is zero, then this value $x = k$ is indeed a real zero of the polynomial.

- Perform a division using $x - k$ as the divisor if using long division, or $x = k$ if using synthetic division. If the remainder is zero, then $x = k$ is indeed a real zero of the polynomial.

Example: Find the zeroes of $f(x) = x^4 - x^3 + x^2 - 3x - 6$

Solution: We want to solve for *x* when $f(x) = x^4 - x^3 + x^2 - 3x - 6 = 0$

The possible real zeroes are

$$\frac{Factors\ of\ constant\ term,\ a_0}{Factors\ of\ leading\ coefficient, a_n}$$

$$= \frac{\pm 1, \pm 2, \pm 3, \pm 6}{\pm 1}$$

$$= \pm 1, \pm 2, \pm 3, \pm 6$$

Let us try and see whether $x = 1$ is a zero of the polynomial. We can perform a synthetic division:

$$\begin{array}{r|rrrrr} 1 & 1 & -1 & 1 & -3 & -6 \\ & & 1 & 0 & 1 & -2 \\ \hline & 1 & 0 & 1 & -2 & -8 \end{array}$$

Remainder $\neq 0$

Since the remainder is not zero, this shows that $x = 1$ is not a zero of the polynomial.

Let us now try and see whether $x = -1$ is a zero of the polynomial. We perform a synthetic division:

-1	1	-1	1	-3	-6
		-1	2	-3	6
	1	-2	3	-6	0

Remainder $= 0$

The remainder is zero, indicating that $x = -1$ is a zero of the polynomial. This also indicates that $x + 1$ is a factor of the polynomial. And from the division above, we can deduce that

$$f(x) = x^4 - x^3 + x^2 - 3x - 6$$

$$= (x + 1)(x^3 - 2x^2 + 3x - 6)$$

Now, let us check and see whether $x = 2$ is a zero of the quotient $x^3 - 2x^2 + 3x - 6$, we perform a synthetic division on it.

2	1	-2	3	-6
		2	0	6
	1	0	3	0

Remainder $= 0$

As the remainder is zero, this indicates that $x = 2$ is a zero of the quotient $x^3 - 2x^2 + 3x - 6$. This also indicates that $x - 2$ is a factor of this quotient.

Therefore, we conclude that when

$$f(x) = x^4 - x^3 + x^2 - 3x - 6 = 0$$

$$\Rightarrow \qquad (x + 1)(x^3 - 2x^2 + 3x - 6) = 0$$

$$\Rightarrow \qquad (x + 1)(x - 2)(x^2 + 3) = 0$$

$$\Rightarrow \qquad x + 1 = 0,\ x - 2 = 0 \text{ or } x^2 + 3 = 0$$

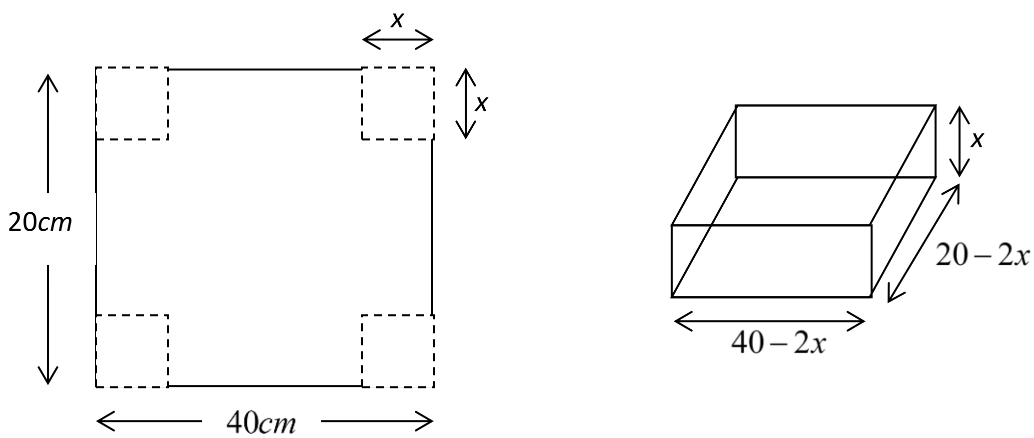
$$\Rightarrow \qquad x = -1,\ x = 2 \text{ or } x^2 = -3$$

But since x^2 cannot be negative, the only real zeroes of the polynomial are

$$x = -1 \text{ or } x = 2$$

Application example: An open box with a volume of $1500\ cm^3$ is to be constructed by taking a piece of cardboard 20 *cm* by 40 *cm*, cutting squares of side length *x cm* from each corner and folding up the sides. Show that this can be done in two different ways and find the exact dimensions of the box in each case.

Solutions: The diagram



The volume of the box is

$$(40 - 2x)(20 - 2x)(x) = 1500$$

For the box to be defined, we need all the dimensions to be non-zero, i.e.

$x > 0$ and $40 - 2x > 0$ and $20 - 2x > 0$

$\Rightarrow$ $x > 0$ and $2x < 40$ and $2x < 20$

$\Rightarrow$ $x > 0$ and $x < 20$ and $x < 10$

$\Rightarrow$ $0 < x < 10$; Combining all the above three conditions gives us the domain.

Continuing, $(40 - 2x)(20 - 2x)(x) = 1500$

$\Rightarrow$ $(2)(20 - x)(2)(10 - x)(x) = 1500$

$\Rightarrow$ $(x)(20 - x)(10 - x) = \frac{1500}{(2)(2)} = 375$

$\Rightarrow$ $(x)(200 - 30x + x^2) = 375$

$\Rightarrow$ $200x - 30x^2 + x^3 = 375$

$\Rightarrow$ $x^3 - 30x^2 + 200x - 375 = 0$

Factors of -375 are ± 3, ± 5, ± 15, ± 25, ± 75, ± 125 and ± 375.
Factors of 1 are ± 1.

Hence, possible roots of the above equation are

$$\frac{Factors\ of\ constant\ term,\ a_0}{Factors\ of\ leading\ coefficient, a_n}$$

$$= \frac{\pm 3, \pm 5, \pm 15, \pm 25, \pm 75, \pm 125, \pm 375}{\pm 1}$$

$$= \pm 3,\ \pm 5,\ \pm 15,\ \pm 25,\ \pm 75,\ \pm 125,\ \pm 375$$

Let's try $x = 3$.

$$\begin{array}{c|cccc} 3 & 1 & -30 & 200 & -375 \\ & & 3 & -81 & 357 \\ \hline & 1 & -27 & 119 & -18 \end{array}$$

Remainder $\neq 0$

Since the remainder when $x = 3$ is not zero, we conclude that $x = 3$ is not a solution.

Now let's try $x = 5$

$$\begin{array}{c|cccc} 5 & 1 & -30 & 200 & -375 \\ & & 5 & -125 & 375 \\ \hline & 1 & -25 & 75 & 0 \end{array}$$

Remainder $= 0$

Since the remainder when $x = 5$ is zero, we conclude that $x = 5$ is a solution.

And therefore, $x - 5$ is a factor of the polynomial. And the above synthetic division tells us that

$$x^3 - 30x^2 + 200x - 375 = (x - 5)(x^2 - 25x + 75)$$

And when

$$x^3 - 30x^2 + 200x - 375 = 0$$

$$\Rightarrow \quad (x - 5)(x^2 - 25x + 75) = 0$$

$$\Rightarrow \quad x - 5 = 0 \quad \text{or} \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad x = 5 \quad \text{or} \quad x = \frac{-(-25) \pm \sqrt{(-25)^2 - 4(1)(75)}}{2(1)}$$

$$\Rightarrow \quad x = 5 \quad \text{or} \quad x = \frac{25 \pm \sqrt{325}}{2} = \frac{25 \pm \sqrt{(25)(13)}}{2} = \frac{25 \pm 5\sqrt{13}}{2}$$

$\Rightarrow \quad x = 5 \quad$ or $\quad x = \frac{25-5\sqrt{13}}{2} \quad$ or $\quad x = \frac{25+5\sqrt{13}}{2}$

But we can see that $\frac{25+5\sqrt{13}}{2} = 12.5 + 2.5 * \sqrt{13} > 10$ which means it is outside the domain of the box. Hence, we cannot accept $x = \frac{25+5\sqrt{13}}{2}$ as a solution.

So the only two possible solutions are $\quad x = 5 \quad$ or $\quad x = \frac{25-5\sqrt{13}}{2} = 12.5 - 2.5\sqrt{13}$

$\therefore$ The dimensions of the box are

(1) When $Height = x = 5\ cm,$

$Length = 40 - 2x = 40 - 2(5) = 30\ cm$

$Width = 20 - 2x = 20 - 2(5) = 10\ cm$

(2) When $Height = x = 12.5 - 2.5\sqrt{13} = 2.5\left(5 - \sqrt{13}\right)\ cm$

$Length = 40 - 2x = 40 - 2\left(12.5 - 2.5\sqrt{13}\right) = 40 - 25 + 5\sqrt{13} = 15 + 5\sqrt{13} = 5\left(3 + \sqrt{13}\right)\ cm$

$Width = 20 - 2x = 20 - 2\left(12.5 - 2.5\sqrt{13}\right) = 20 - 25 + 5\sqrt{13} = 5\sqrt{13} - 5 = 5\left(\sqrt{13} - 1\right)\ cm$

3.5 Rational Functions

All rational functions $f(x) = \frac{P(x)}{Q(x)}$, provided that $\frac{P(x)}{Q(x)}$ is irreducible,

i. Can be written as $\frac{P(x)}{Q(x)} = s(x) + \frac{t(x)}{Q(x)}$

ii. Have asymptotes.

- The vertical asymptotes are given by the solutions to $Q(x) = 0$
- The non-vertical asymptote is given by $y = s(x)$.

A rational function will have at the very least one non-vertical asymptote.

Examples:

[1] $y = \frac{1}{x}$

This can be written as $\frac{1}{x} = 0 + \frac{1}{x}$

- The vertical asymptote is $x = 0$

- The non-vertical asymptote is $y = 0$

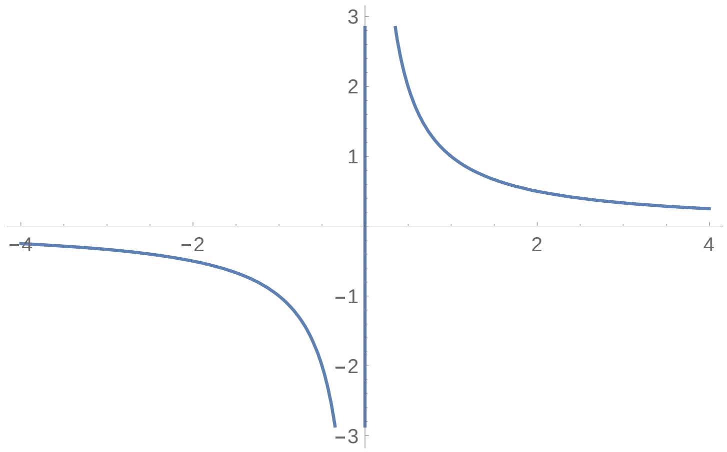
To find where the parts of the graph lie, we generate a value table. The domain is divided by the vertical asymptotes and where the graph intersects the non-vertical asymptote, if any:

Interval	$\boldsymbol{x < 0}$	$\boldsymbol{x > 0}$
x-coordinate test value	$x = -1$	$x = 1$
Value of $y = \frac{1}{x}$	$\frac{1}{-1} = -1$	$\frac{1}{1} = 1$

We conclude that:

- The graph to the left of the vertical asymptote lies below the non-vertical asymptote.
- The graph to the right of the vertical asymptote lies above the non-vertical asymptote.

Graph of $y = \frac{1}{x}$:



[2] $y = \frac{1}{x^2}$

This can be written as $\frac{1}{x^2} = 0 + \frac{1}{x^2}$

- The vertical asymptote is the solution to $x^2 = 0 \Rightarrow x = 0$

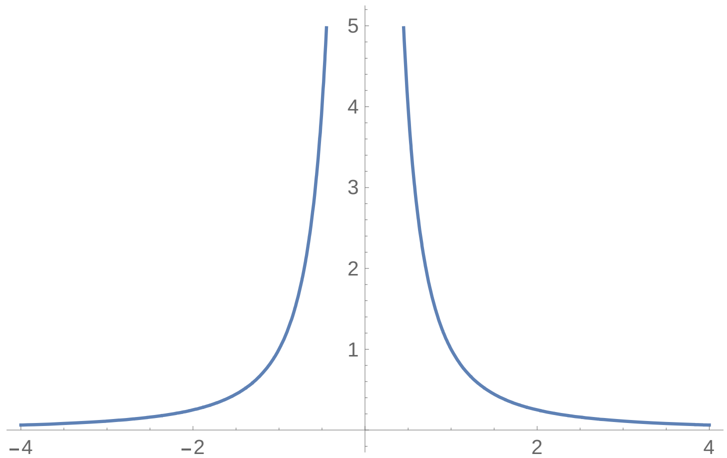
- The non-vertical asymptote is $y = 0$

To find where the parts of the graph lie, we generate a value table. The domain is divided by the vertical asymptotes, and where the graph intersects the non-vertical asymptote, if any:

Interval	$\boldsymbol{x < 0}$	$\boldsymbol{x > 0}$
x-coordinate test value	$x = -1$	$x = 1$
Value of $y = \frac{1}{x^2}$	$\frac{1}{(-1)^2} = 1$	$\frac{1}{1^2} = 1$

We conclude that:
- The graph to the left of the vertical asymptote lies above the non-vertical asymptote.
- The graph to the right of the vertical asymptote lies above the non-vertical asymptote.

Graph of $y = \frac{1}{x^2}$:



[3] $y = \frac{3x+5}{x+2}$

This rational function is an improper rational. So we perform a division

$$\begin{array}{r|rr} -2 & 3 & 5 \\ & & -6 \\ \hline & 3 & -1 \end{array}$$

Thus, $\frac{3x+5}{x+2} = 3 + \frac{-1}{x+2}$

- The vertical asymptote is the solution to $x + 2 = 0 \Rightarrow x = -2$
- The non-vertical asymptote is $y = 3$.

To find out whether the graph intersects the non-vertical asymptote, we equate the asymptote to the function:

$$\frac{3x+5}{x+2} = 3$$
$$\Rightarrow \quad 3 + \frac{-1}{x+2} = 3$$
$$\Rightarrow \quad \frac{-1}{x+2} = 0 \text{ which is impossible.}$$

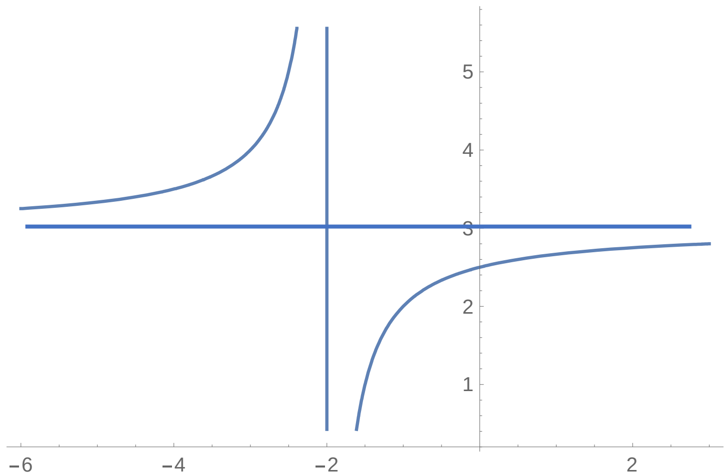
Hence, the graph of the function does not intersect the non-vertical asymptote.

Interval	$\boldsymbol{x < -2}$	$\boldsymbol{x > -2}$
x-coordinate test value	$x = -3$	$x = 0$
Value of $y = \frac{3x+5}{x+2}$	$y = 4$	$y = 2.5$

We conclude that

- The graph to the left of the vertical asymptote lies above the non-vertical asymptote.
- The graph to the right of the vertical asymptote lies below the non-vertical asymptote.

Graph of $y = \frac{3x+5}{x+2} = 3 + \frac{-1}{x+2}$



[4] $\quad y = \frac{2x^2+7x-4}{x^2+x-2}$

This is an improper rational, so we perform a division:

					2				
x^2	$+$	x	$-$	2	$2x^2$	$+$	$7x$	$-$	4
					$2x^2$	$+$	$2x$	$-$	4
							$5x$	$+$	0

Hence, $y = \frac{2x^2+7x-4}{x^2+x-2} = 2 + \frac{5x}{x^2+x-2}$

- The vertical asymptotes are the solutions to $x^2 + x - 2 = 0$

  $\Rightarrow \quad (x+2)(x-1) = 0$

  $\Rightarrow \quad x + 2 = 0$ or $x - 1 = 0$

  $\Rightarrow \quad x = -2$ or $x = 1$

- The non-vertical asymptote is $y = 2$

To find out whether the graph intersects the non-vertical asymptote, we equate the function with the non-vertical asymptote, i.e.

$$2 = 2 + \frac{5x}{x^2+x-2}$$
$$\Rightarrow \quad \frac{5x}{x^2+x-2} = 0$$
$$\Rightarrow \quad 5x = 0$$
$$\Rightarrow \quad x = 0$$

The graph intersects the non-vertical asymptote at $x = 0$

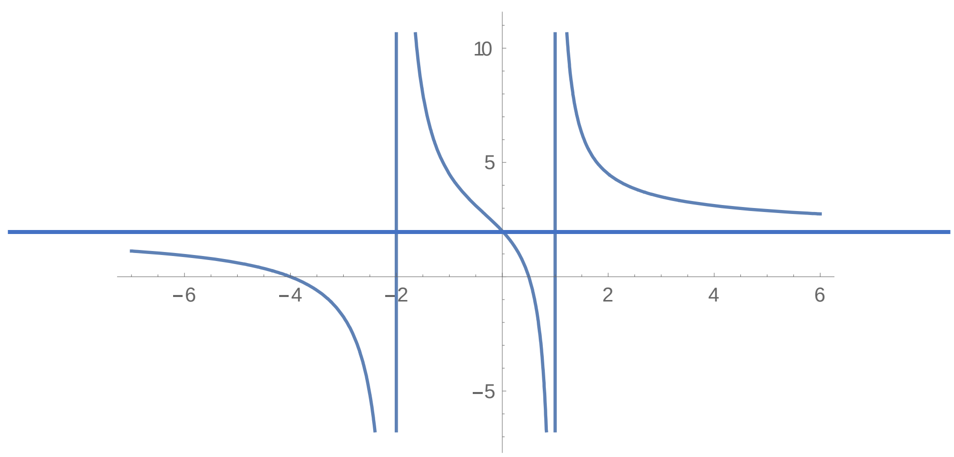
We generate the value table as follows:

Interval	$x < -2$	$-2 < x < 0$	$0 < x < 1$	$x > 1$
x-coordinate test value	$x = -3$	$x = -1$	$x = 0.5$	$x = 2$
Value of $y = \dfrac{2x^2 + 7x - 4}{x^2 + x - 2}$	$-\dfrac{7}{4} = -1\dfrac{3}{4}$	$\dfrac{9}{2} = 4\dfrac{1}{2}$	0	$\dfrac{18}{4} = 4\dfrac{1}{2}$

We conclude that

- The graph to the left of the vertical asymptote $x = -2$ lies below the non-vertical asymptote.
- The graph slightly to the right of the vertical asymptote $x = -2$ lies above the non-vertical asymptote.
- The graph slightly to the left of the vertical asymptote $x = 1$ lies below the non-vertical asymptote.
- The graph to the right of the vertical asymptote $x = 1$ lies above the non-vertical asymptote.

Graph of $y = \frac{2x^2+7x-4}{x^2+x-2} = 2 + \frac{5x}{x^2+x-2}$:



[5] $f(x) = \frac{2x-4}{x^2+x+1}$

$\Rightarrow \quad f(x) = 0 + \frac{2x-4}{x^2+x+1}$

- The non-vertical asymptote is $y = 0$
- The vertical asymptote is given by the solutions to $x^2 + x + 1 = 0$. However, its discriminant

$$b^2 - 4ac = 1^2 - 4(1)(1) = -ve$$

Indicates that $x^2 + x + 1 = 0$ has no solution. Hence, the rational has no vertical asymptote.

To find out whether the graph intersects the non-vertical asymptote, we equate the asymptote with the function:

$\frac{2x-4}{x^2+x+1} = 0$

$\Rightarrow \quad 2x - 4 = 0$

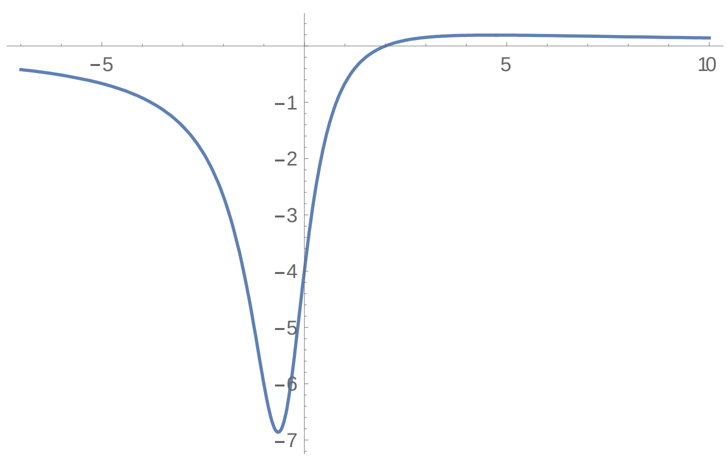
$\Rightarrow \quad 2x = 4$

$\Rightarrow \quad x = \frac{4}{2} = 2$

Interval	$x < 2$	$x > 2$
x-coordinate test value	$x = 0$	$x = 3$
Value of $y = \frac{2x-4}{x^2+x+1}$	$y = -4$	$y = \frac{2}{13}$

We conclude that
- The graph to the left of the *x*-intercept is below the non-vertical asymptote.
- The graph to the right of the *x*-intercept is above the non-vertical asymptote.

Graph of $r(x) = \frac{2x-4}{x^2+x+1} = 0 + \frac{2x-4}{x^2+x+1}$



[6] $f(x) = \frac{x^3+3x^2}{x^2-4} = \frac{x^3+3x^2+0x+0}{x^2+0x-4}$

This is an improper rational so we perform a division,

$$\begin{array}{r|rrrrrrr} & x & + & 3 & & & & \\ \hline x^2 + 0x - 4 & x^3 & + & 3x^2 & + & 0x & + & 0 \\ & x^3 & + & 0x^2 & - & 4x & & \\ \hline & & & 3x^2 & + & 4x & + & 0 \\ & & & 3x^2 & + & 0x & - & 12 \\ \hline & & & & & 4x & + & 12 \end{array}$$

Hence, $r(x) = \frac{x^3+3x^2}{x^2-4} = x + 3 + \frac{4x+12}{x^2-4}$

- The non-vertical asymptote is $y = x + 3$

- The vertical asymptotes are the solutions to $x^2 - 4 = 0$
  $\Rightarrow$ $x^2 = 4$
  $\Rightarrow$ $x = 2$ or $x = -2$

To find out whether the graph intersects the non-vertical asymptote, we equate the non-vertical asymptote with the function:

$x + 3 + \frac{4x+12}{x^2-4} = x + 3$

$\Rightarrow$ $\frac{4x+12}{x^2-4} = 0$

$\Rightarrow$ $4x + 12 = 0$

$\Rightarrow$ $4x = -12$

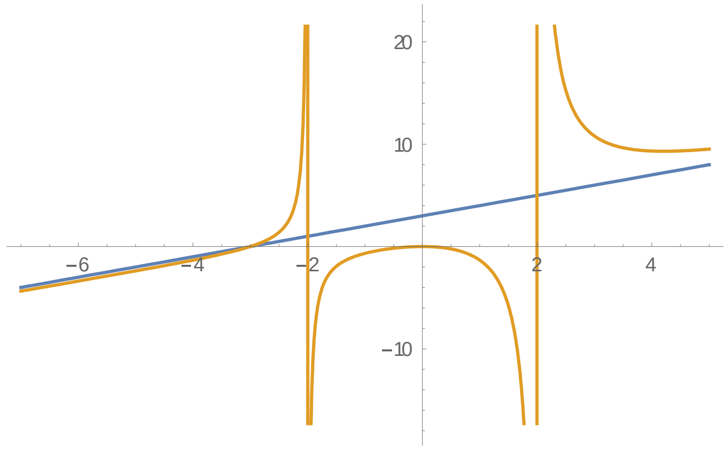
$\Rightarrow$ $x = \frac{12}{4} = -3$

Interval	$x < -3$	$-3 < x < -2$	$-2 < x < 2$	$x > 2$
x-coordinate test value	$x = -4$	$x = -2.5$	$x = 0$	$x = 3$
Value of $y = \frac{x^3 + 3x^2}{x^2 - 4}$	$y = \frac{-16}{12} = -\frac{4}{3} = -1\frac{1}{3}$	$y = \frac{3.125}{2.25} \approx 13$	$y = 0$	$y = \frac{54}{5} = 10\frac{4}{5}$

We conclude that

- The graph to the left of the non-vertical asymptote intercept lies below the asymptote.
- The graph between the non-vertical asymptote intercept and the vertical asymptote $x = -2$ lies above the non-vertical asymptote.
- The graph between the two vertical asymptotes lies below the non-vertical asymptote.
- The graph to the right of the vertical asymptote $x = 2$ lies above the non-vertical asymptote.

The graph of $y = \frac{x^3+3x^2}{x^2-4} = x + 3 + \frac{4x+12}{x^2-4}$:



[7] $f(x) = \frac{x^3-2x^2+16}{x-2} = \frac{x^3-2x^2+0x+16}{x-2}$

We perform a division

2	1	-2	0	16
		2	0	0
	1	0	0	16

Hence, $y = \frac{x^3-2x^2+16}{x-2} = x^2 + 0x + 0 + \frac{16}{x-2}$

$\Rightarrow \quad y = x^2 + \frac{16}{x-2}$

- The non-vertical asymptote is $y = x^2$
- The vertical asymptote is the solution to $x - 2 = 0 \Rightarrow x = 2$

Now, we equate the non-vertical asymptote with the function to see if the graph intersects the non-vertical asymptote,

$$x^2 = x^2 + \frac{16}{x-2}$$

$\Rightarrow \quad \frac{16}{x-2} = 0$ which is impossible.

Hence, the graph does not intersect the non-vertical asymptote.

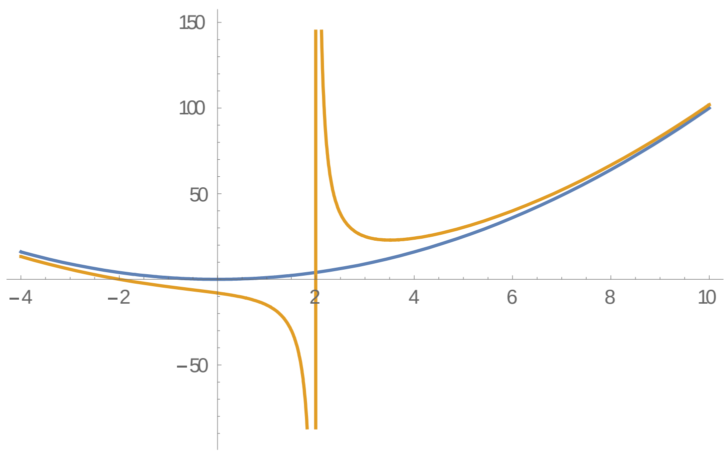
We now generate a value table:

Interval	$x < 2$	$x > 2$
x-coordinate test value	$x = 0$	$x = 3$
Value of $y = \frac{x^3-2x^2+16}{x-2}$	$y = \frac{16}{-2} = -8$	$y = 25$

We conclude that

- The graph to the left of the vertical asymptote lies below the non-vertical asymptote.
- The graph to the right of the vertical asymptote lies above the non-vertical asymptote.

Graph of $y = \frac{x^3-2x^2+16}{x-2} = x^2 + \frac{16}{x-2}$



4.1 Exponential Functions

Definition of an Exponential Function:

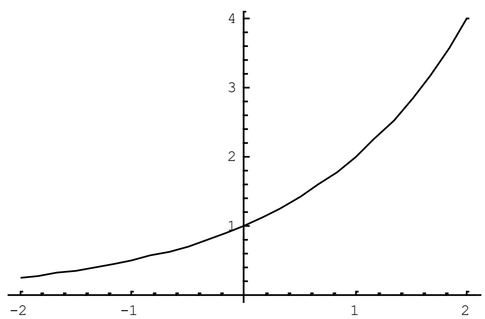
$y = a^x$

- Where the base must be positive, i.e. $a > 0$
- Its domain is $-\infty < x < \infty$ or $x \in (-\infty, \infty)$
- And its range is $y > 0$ or $y \in (0, \infty)$

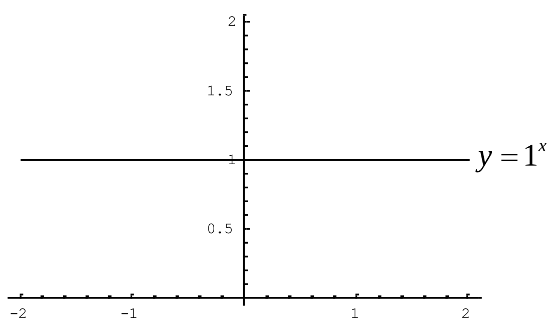
Graphs of Exponential Functions, $y = a^x$

Three possibilities:

i. When $a > 1$.

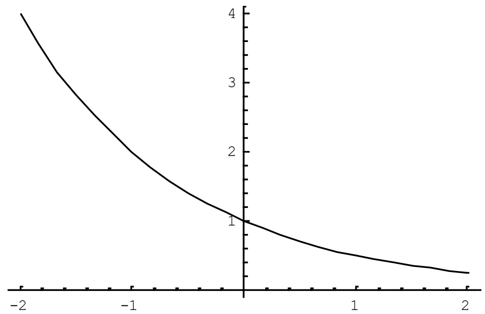


ii. When $a = 1$



Note: In this situation, we have $y = 1^x = 1$ which is basically a constant function.

iii. When $0 < a < 1$



Observations:

i. The graph of $y = a^x$ intersects the *y*-axis at $(0,1)$.

ii. The exponential graph is always above the *x*-axis, i.e. no matter what value we assign to the variable *x*,

$$a^x > 0$$

Thus, its range is $y \in (0, \infty)$

iii. In the graph of $y = a^x$, where $a > 0$ and $a \neq 1$, we have a horizontal asymptote $y = 0$, i.e. the *x*-axis is the horizontal asymptote of the exponential function.

iv. The function $y = a^x$, where $a > 0$ and $a \neq 1$, is a one-to-one function on its entire domain $x \in (-\infty, \infty)$, i.e we can define its inverse function.
(We shall discuss the inverse of the exponential function in Section 4.3 - Logarithmic Functions.)

Another observation:

Let us consider the base $a > 1$. And let us also consider a function

$$y = a^{-x}$$

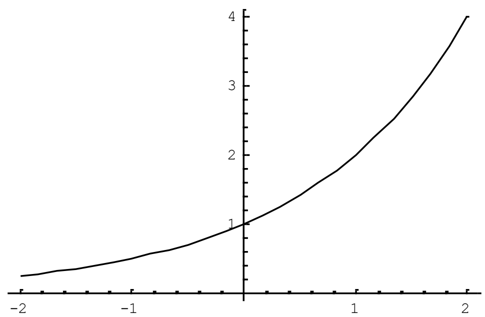
$\Rightarrow$ $y = \frac{1}{a^x}$

$\Rightarrow$ $y = \frac{1^x}{a^x}$, We know that $1^x = 1$

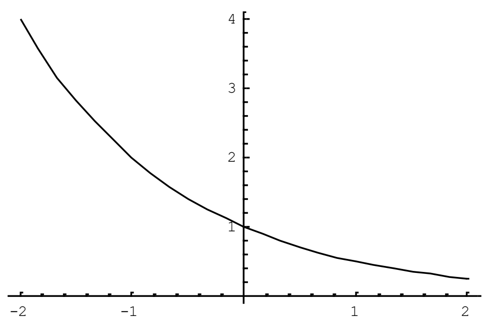
$\Rightarrow$ $y = \left(\frac{1}{a}\right)^x$

$\therefore \qquad y = a^{-x} = \left(\frac{1}{a}\right)^x$

The graph of $y = a^x$, where $a > 1$, is

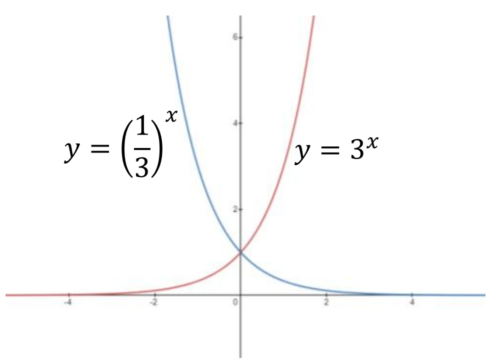


The graph of $y = a^{-x}$, where $a > 1$, is obtained by reflecting the graph of $y = a^x$ about the *y*-axis,



In summary, the graph of $y = a^x$ and the graph of $y = \left(\frac{1}{a}\right)^x$ are reflections of each other about the *y*-axis.

Example: The graph of $y = 3^x$ and the graph of $y = \left(\frac{1}{3}\right)^x$ are reflections of each other about the *y*-axis.



Compound Interest

Let us say we deposit an amount *P* in a savings account. This account earns interest at an annual interest rate *r* compounded *n* times per year. After *t* years, the amount in the account grows to

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

Where $A(t) = amount\ after\ t\ years$
$P = principal$
$r = interest\ rate\ per\ year$
$n = number\ of\ times\ interest\ is\ compounded\ per\ year$
And $t = number\ of\ years$

Annual Percentage Yield (Effective Interest Rate)

Suppose we deposit money into a savings account that returns an earning of \$*X*. The Annual Percentage Yield (*AYL*) is the interest rate that would return the same amount of earning, i.e. \$*X*, if we had applied this *AYL* to the principal amount compounded once a year.

Examples: A bacteria culture contains 1,500 bacteria initially and doubles every hour.
[a] Find a function that models the number of bacteria after *t* hours.
[b] Find the number of bacteria after 24 hours.

Solution: Formula $A(t) = P(1 + i)^t$

Where $A(t) = number\ of\ bacteria\ after\ t\ hours$,
$1 + i = 2$, the number of bacteria doubles every *t* hours,
and $P = 1500\ bacteria$.

[a] $A(t) = 1500(2)^t$

[b] $A(24) = 1500(2)^{24}$

Examples on Compound Interest:

[1] If \$500 is invested at an interest rate of 3.75% per year, compounded quarterly, find the value of the investment after 2 years.

Solution: Formula $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$

Where $P = \$500$,
$r = 3.75\% = 0.0375$, We have to use the actual number, not percentage.
$n = 4$ There are 4 times of compounds in a year.
and $t = 2\ years$.

$\therefore \qquad A(2) = 500\left(1 + \frac{0.0375}{4}\right)^{(4)(2)}$

$$= 500(1 + 0.009375)^8$$

$$= \$500(1.009375)^8$$

[2] Find the present value of \$100,000 if interest is paid at a rate of 8% per year, compounded monthly, for 5 years.

Solution: Formula $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$

Where $A(5) = \$100{,}000$,
$r = 8\% = 0.08$, We have to use the actual number, not percentage.
$n = 12$ There are 12 times of compounds in a year.
and $t = 5\ years$.

We want to find the present value or the principal, *P*, that must be invested now to produce an amount of \$100,000 five years from now.

$\therefore \qquad 100{,}000 = P\left(1 + \frac{0.08}{12}\right)^{(12)(5)}$

$$= P\left(1 + \frac{8}{1200}\right)^{60}$$

$$= P\left(1 + \frac{1}{150}\right)^{60}$$

$$= P\left(\frac{151}{150}\right)^{60}$$

$\Rightarrow \qquad P = \frac{100000}{\left(\frac{151}{150}\right)^{60}}$

$$= \$100000\left(\frac{150}{151}\right)^{60} \approx \$67{,}121.04$$

[3] Find the annual percentage yield for an investment that earns 8% per year, compounded monthly.

Solution: Let $r_{eff} = Annual\ Percentage\ Yield$

So we let $1 + r_{eff} = \left(1 + \frac{r}{n}\right)^n$, since we want the yield after $t = 1$ year.

$\Rightarrow \qquad r_{eff} = \left(1 + \frac{r}{n}\right)^n - 1$

Where $r = 8\% = 0.08,$

$n = 12$ Compounded monthly, and there are 12 months in a year.

$$\Rightarrow \qquad r_{eff} = \left(1 + \frac{0.08}{12}\right)^{12} - 1$$

$$= \left(1 + \frac{1}{150}\right)^{12} - 1$$

$$= \left(\frac{151}{150}\right)^{12} - 1 \approx 0.083 \approx 8.3\%$$

You would in actual fact be receiving an interest rate of 8.3% under this agreement.

[4] A couple has a 6-year-old child who will be ready for college in 12 years. The couple estimates that $85,000 will be needed to pay for the estimated four years of college. To the nearest dollar, how much would have to be invested now at a nominal rate of 10%, compounded monthly, to meet this need?

Solution: Formula $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$

Where $A(12) = \$85{,}000,$
$r = 10\% = 0.1,$ We have to use the actual number, not percentage.
$n = 12$ There are 12 compounds in a year.
and $t = 12\ years.$

We want to find the present value or the principal, *P*, that must be invested now to produce an amount of $85,000 twelve years from now.

$$\therefore \qquad 85{,}000 = P\left(1 + \frac{0.1}{12}\right)^{(12)(12)}$$

$$= P\left(1 + \frac{1}{120}\right)^{144}$$

$$= P\left(\frac{121}{120}\right)^{144}$$

$$\Rightarrow \qquad P = \frac{85000}{\left(\frac{121}{120}\right)^{144}}$$

$$= \$85000\left(\frac{120}{121}\right)^{144} \approx \$25{,}729$$

4.2 The Natural Exponential Function

The Number *e*

Definition: $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

OR $e = \lim_{n\to 0}(1+n)^{\frac{1}{n}}$

$\therefore$ $e = 2.7182818284590452353\cdots$

It is an irrational number, i.e. the decimal digits go on indefinitely with no obvious pattern. And it cannot be written as an exact fraction.

The Natural Exponential Function

Definition: $f(x) = e^x$, the base of this exponential function is the number *e*.

Continuously Compounded Interest

If an investment starts generating an income for us the moment we put in the investment, this is said to be a continuously compounded interest. We can think of it as having the interest compounded an infinitely huge number of times per year!

The continuously compounded interest formula is

$$A(t) = P\left(\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt}\right)$$

Let $m = \frac{r}{n}$

$\Rightarrow$ $n = \frac{r}{m}$

And as $n \to \infty$, we have $\frac{r}{n} \to 0$ and $m \to 0$. Continuing,

$$A(t) = P\left(\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt}\right)$$

$$= P\left(\lim_{m\to 0}(1+m)^{\left(\frac{r}{m}\right)(t)}\right)$$

$$= P\left(\lim_{m\to 0}(1+m)^{\frac{1}{m}}\right)^{rt}$$

$\therefore$ $A(t) = Pe^{rt}$, since $e = \lim_{m\to 0}(1+m)^{\frac{1}{m}}$

Where $A(t) = amount\ after\ t\ years$
$P = principal$
$r = interest\ rate\ per\ year$
And $t = number\ of\ years$

Example: Stavros buys a brand new car today at the price of \$15,000. It is estimated that the car depreciates in value at the continuous rate of 8% per year. How much will the car be worth exactly 5 years from now?

Solution: Formula $A(t) = Pe^{rt}$

Where $P = \$15{,}000$

$r = -8\% = -0.08$, negative value indicates a depreciation or a decrease in value.

And $t = 5\ years.$

$\therefore$ $A(8) = 15000e^{(-0.08)(5)}$

$= \$15000e^{-0.4} \approx \$10{,}054.80$

4.3 Logarithmic Functions

The logarithmic function and the exponential function are inverses of each other, i.e.

$$\log_a x = y \quad \Leftrightarrow \quad a^y = x$$

base *exponent*

Examples:

$$2^3 = 8 \quad \Leftrightarrow \quad \log_2 8 = 3$$

$$3^{-2} = \tfrac{1}{9} \quad \Leftrightarrow \quad \log_3 \tfrac{1}{9} = -2$$

$$64^{\frac{1}{3}} = 4 \quad \Leftrightarrow \quad \log_{64} 4 = \tfrac{1}{3}$$

$$125^{\frac{2}{3}} = 25 \quad \Leftrightarrow \quad \log_{125} 25 = \tfrac{2}{3}$$

- The base of the logarithmic function must be positive and cannot be 1 i.e.

$$0 < a < 1 \text{ or } a > 1$$

- The domain of the logarithmic function must be positive, i.e.

$$x > 0$$

- The range of the logarithmic function is $-\infty < y < \infty$

Properties of the Logarithmics

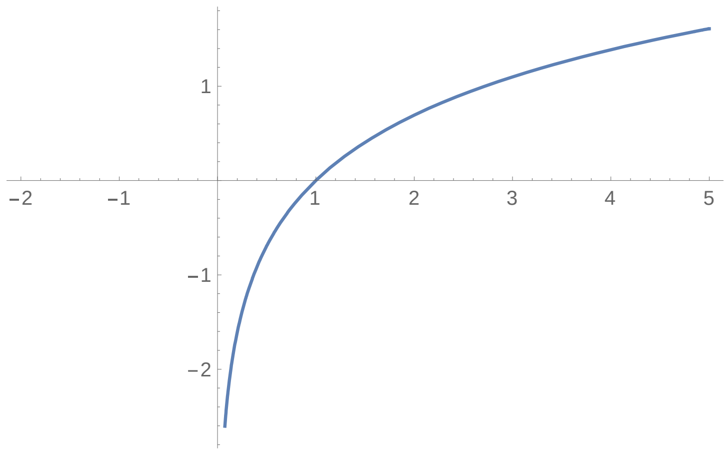
1. $\log_a 1 = 0$ because $a^0 = 1$
2. $\log_a a = 1$ because $a^1 = a$
3. $\log_a(mn) = \log_a m + \log_a n$
4. $\log_a \frac{m}{n} = \log_a m - \log_a n$
5. $\log_a m^r = r \log_a m$
6. $\log_a a^x = x$
7. $a^{\log_a x} = x$

8. If $\log_a x = \log_a y$, then $x = y$

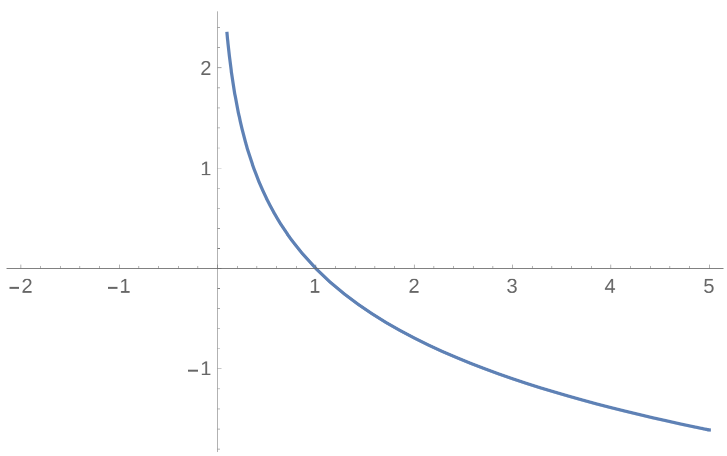
Graphs of the Logarithmic Functions, $y = \log_a x$

Two possibilities:

1. When $a > 1$



2. When $0 < a < 1$



Some observations:

1. We can also show that the graph of $y = \log_a x$ and the graph of $y = \log_{\frac{1}{a}} x$ are reflections of each other about the x-axis.

2. Since $y = a^x$ and $y = \log_a x$ are inverses of each other, the graph of $y = a^x$ and the graph of $y = \log_a x$ are reflections of each other about the diagonal line $y = x$.

Common Logarithms

A logarithm with base 10 is called a common logarithm. In the American system, a logarithm with base 10 can be written with the base omitted, i.e.

$$\log_{10} x = \log x$$

Natural Logarithmics

A logarithm with base $e = 2.718281828459 \cdots$ is called a natural logarithm. A logarithm with base *e* can be written as follows:

$$\log_e x = \ln x$$

Remember that $\ln x$ is the shortcut way of writing the longer $\log_e x$.

Applications

1. A total of $\$P$ is invested at an annual interest rate of 9% annually. Assuming that the interest is compounded continuously, how long will it take for the amount of money to double?

Solution: If the interest is compounded continuously, then the model is

$A = Pe^{rt}$

$\Rightarrow$ $2P = Pe^{rt}$, since we want the initial amount *P* to double or to become $2P$

$\Rightarrow$ $2 = e^{rt}$

$\Rightarrow$ $rt = \log_e 2 = \ln 2$

$\Rightarrow$ $t = \frac{\ln 2}{r}$

Given that $r = 9\% = 0.09$ per year,

$t = \frac{\ln 2}{0.09}$ years

2. **Radioactive Decay**

Given a radioactive material of initial mass m_0 with a half-life $t_{\frac{1}{2}}$. The amount remaining after *t* years is given by

$$m = m_0 \left(\frac{1}{2}\right)^{\frac{t}{t_{\frac{1}{2}}}}$$

Rewrite the above formula as *t* in terms of *m*.

Answer: $m = m_0 \left(\frac{1}{2}\right)^{\frac{t}{t_{\frac{1}{2}}}}$

$\Rightarrow \quad \frac{m}{m_0} = \left(\frac{1}{2}\right)^{\frac{t}{t_{\frac{1}{2}}}}$

$\Rightarrow \quad \frac{t}{t_{\frac{1}{2}}} = \log_{\frac{1}{2}} \frac{m}{m_0}$

$\Rightarrow \quad t = t_{\frac{1}{2}} \log_{\frac{1}{2}} \frac{m}{m_0}$

3. An employee is hired at a salary of \$20,000 per year. If this employee is given a 10% salary increase each year, how long will it take for the salary to exceed \$48,000 per year?

Solution: Formula $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$

Where $t = 10\; years$

$A(10) > \$48{,}000$, we want the salary to exceed \$48,000

$P = \$20{,}000$

$r = 10\% = 0.1$

And $n = 1$, compounded only once per year.

$\therefore \quad A(10) > 48000$

$\Rightarrow \quad 20000\left(1 + \frac{0.1}{1}\right)^{(1)(t)} > 48000$

$\Rightarrow \quad (1 + 0.1)^t > 48000/20000$

$\Rightarrow \quad (1.1)^t > 2.4$

$\Rightarrow \qquad \ln 1.1^t > \ln 2.4$

$\Rightarrow \qquad t \ln 1.1 > \ln 2.4$

$\Rightarrow \qquad t > \frac{\ln 2.4}{\ln 1.1}$

$\Rightarrow \qquad t > 9.1855 \; years$

$\Rightarrow \qquad t = 10 \; years \; at \; the \; very \; least$

$\therefore$ The employee must work for at least ten years before getting a salary of at least $48,000 per year.

4.4 Laws of Logarithms

Memorise the following laws:

- $\log_a(AB) = \log_a A + \log_a B$
- $\log_a \frac{A}{B} = \log_a A - \log_a B$
- $\log_a A^c = c \log_a A$

Important:

- $\log_a(x + y) \neq \log_a x + \log_a y$
- $\frac{\log_a x}{\log_a y} \neq \log_a \frac{x}{y}$
- $(\log_a A)^c \neq c \log_a A$

Examples:

Evaluate the expressions:

[1] $\log_2 160 - \log_2 5 = \log_2 \frac{160}{5}$

$= \log_2 32 = 5$, since $2^5 = 32$

[2] $\log_{12} 9 + \log_{12} 16 = \log_{12}\big((9)(16)\big)$

$= \log_{12} 144 = 2$, since $12^2 = 144$

[3] $\log_2 8^{33} = 33 \log_2 8$

$= (33)(3) = 99$, since $2^3 = 8$

Expand the expressions:

[4] $\log \frac{a^2}{b^4\sqrt{c}} = \log a^2 - \log b^4 - \log \sqrt{c}$

$= \log a^2 - \log b^4 - \log c^{\frac{1}{2}}$

$= 2\log a - 4\log b - \frac{1}{2}\log c$

[5] $$\log\sqrt{x\sqrt{y\sqrt{z}}} = \log\left(x\sqrt{y\sqrt{z}}\right)^{\frac{1}{2}}$$

$$= \frac{1}{2}\log\left(x\sqrt{y\sqrt{z}}\right)$$

$$= \frac{1}{2}\left(\log x + \log\sqrt{y\sqrt{z}}\right)$$

$$= \frac{1}{2}\left(\log x + \log\left(y\sqrt{z}\right)^{\frac{1}{2}}\right)$$

$$= \frac{1}{2}\left(\log x + \frac{1}{2}\log\left(y\sqrt{z}\right)\right)$$

$$= \frac{1}{2}\log x + \frac{1}{4}\log\left(y\sqrt{z}\right)$$

$$= \frac{1}{2}\log x + \frac{1}{4}\left(\log y + \log\sqrt{z}\right)$$

$$= \frac{1}{2}\log x + \frac{1}{4}\left(\log y + \log z^{\frac{1}{2}}\right)$$

$$= \frac{1}{2}\log x + \frac{1}{4}\left(\log y + \frac{1}{2}\log z\right)$$

$$= \frac{1}{2}\log x + \frac{1}{4}\log y + \frac{1}{8}\log z$$

Combine the expressions:

[6] $$\log_5(x^2-1) - \log_5(x-1) = \log_5\frac{x^2-1}{x-1}$$

$$= \log_5\frac{(x-1)(x+1)}{x-1} = \log_5(x+1)$$

[7] $$\frac{1}{3}\log(x+2)^3 + \frac{1}{2}(\log x^4 - \log(x^2-x-6)^2)$$

$$= \frac{3}{3}\log(x+2) + \frac{1}{2}\log x^4 - \frac{1}{2}\log(x^2-x-6)^2$$

$$= \log(x+2) + \frac{4}{2}\log x - \frac{2}{2}\log(x^2-x-6)$$

$$= \log(x+2) + 2\log x - \log\big((x-3)(x+2)\big)$$

$$= \log(x+2) + 2\log x - (\log(x-3) + \log(x+2))$$

$= \log(x+2) + 2\log x - \log(x-3) - \log(x+2)$

$= 2\log x - \log(x-3)$

$= \log x^2 - \log(x-3)$

$= \log \frac{x^2}{x-3}$

Change of Base Formula

$$\log_b x = \frac{\log_a x}{\log_a b}$$

The change of base formula allows us to evaluate a logarithm of base other than 10 or *e*, by changing the base to either 10 or *e* and then using a calculator to evaluate its value.
The following examples are purely for your interest only, since you are NOT allowed to bring an electronic calculator to your labs and tests.

Evaluate each logarithm correct to 6 decimal places.

(a) $\log_{12} 10 = \dfrac{\ln 10}{\ln 12} \approx 0.926628$

OR $\log_{12} 10 = \dfrac{\log_{10} 10}{\log_{10} 12} \approx 0.926628$

(b) $\log_2 8.4 = \dfrac{\ln 8.4}{\ln 2} \approx 3.070389$

OR $\log_2 8.4 = \dfrac{\log_{10} 8.4}{\log_{10} 2} \approx 3.070389$

Example: Simplify $(\log_2 5)(\log_5 7)$.

Solution: $(\log_2 5)(\log_5 7)$

$= \left(\frac{\log_5 5}{\log_5 2}\right)(\log_5 7)$

$= \left(\frac{1}{\log_5 2}\right)\log_5 7$

$$= \frac{\log_5 7}{\log_5 2} = \log_2 7$$

The above example is a verification of the following formula:

$$(\log_a b)(\log_b c) = \log_a c$$

4.5 Exponential and Logarithmic Equations

Guidelines for Solving Exponential Equations

1. Isolate the exponential expression on one side of the equation.
2. Take the logarithm of each side, then use the Laws of Logarithms to bring down the exponent.
3. Solve for the variable.

Guidelines for Solving Logarithmic Equations

1. Combine the logarithmic terms and then isolate the resulting logarithmic term on one side of the equation.
2. Write the equation in exponential form.
3. Solve for the variable.

Examples on solving exponential and logarithmic equations:

Determine the *x*-value of each equation:

1. $4^{2x-7} = 64$

$\Rightarrow$ $4^{2x-7} = 4^3$; Knowing that $64 = 4^3$.

$\Rightarrow$ $2x - 7 = 3$; Applying the One-to-one property for exponentials.

$\Rightarrow$ $2x = 3 + 7 = 10$

$\Rightarrow$ $x = \frac{10}{2} = 5$

2. $\log_2(x+3) = 5$

We must have $x + 3 > 0$ $\Rightarrow$ $x > -3$

Continuing, $\log_2(x+3) = 5$

$\Leftrightarrow$ $x + 3 = 2^5$; Knowing that $a^b = c \Leftrightarrow \log_a c = b$.

$\Rightarrow$ $x + 3 = 32$; Knowing that $2^5 = 32$.

$\Rightarrow$ $x = 32 - 3 = 29$

3. $e^{x^2-3} = e^{x-2}$

$\Rightarrow$ $x^2 - 3 = x - 2$; Applying the One-to-one property for exponentials.

$\Rightarrow$ $x^2 - 3 - x + 2 = 0$

$\Rightarrow \qquad x^2 - x - 1 = 0$

$\Rightarrow \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$; We can see that $a = 1$, $b = -1$ and $c = -1$.

$\Rightarrow \qquad x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times (-1)}}{2 \times 1}$

$\Rightarrow \qquad x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$

$\Rightarrow \qquad x = \frac{1+\sqrt{5}}{2}$ or $x = \frac{1-\sqrt{5}}{2}$

4. $\qquad e^{2x} - 4e^x - 5 = 0$

$\Rightarrow \qquad (e^x)^2 - 4e^x - 5 = 0$

$\Rightarrow \qquad (e^x - 5)(e^x + 1) = 0$

$\Rightarrow \qquad e^x - 5 = 0$ or $e^x + 1 = 0$

$\Rightarrow \qquad e^x = 5$ or $e^x = -1$

But $e^x > 0$, so $e^x = -1$ cannot be accepted. Therefore, the only solution is from

$\qquad e^x = 5$

$\Rightarrow \qquad x = \ln 5$

5. $\qquad \ln x - \ln(x+1) = 2$

We must have $x > 0$ and $x + 1 > 0$

$\Rightarrow \qquad x > 0$ and $x > -1$

$\Rightarrow \qquad x > 0$

Continuing, $\qquad \ln x - \ln(x+1) = 2$

$\Rightarrow \qquad \ln \frac{x}{x+1} = 2$

$\Rightarrow \qquad \frac{x}{x+1} = e^2$

$\Rightarrow \qquad x = e^2(x+1)$

$\Rightarrow \qquad x = e^2x + e^2$

$\Rightarrow \qquad x - e^2x = e^2$

$\Rightarrow \qquad (1 - e^2)x = e^2$

$\Rightarrow \qquad x = \frac{e^2}{1-e^2}$

But $\frac{e^2}{1-e^2} < 0$, which means $x < 0$. We cannot assign this value to *x*, since we require $x > 0$.

Therefore, $\ln x - \ln(x + 1) = 2$ has no solution.

6. $\qquad \log(8x) - \log(1 + \sqrt{x}) = 2$

We must have $8x > 0$ and $1 + \sqrt{x} > 0$. Also, from $\sqrt{x}$, we must have $x \geq 0$.

Combining the above two conditions, we conclude that we must have $x > 0$.

Continuing, $\qquad \log(8x) - \log(1 + \sqrt{x}) = 2$

$\Rightarrow \qquad \log\frac{8x}{1+\sqrt{x}} = 2$

$\Rightarrow \qquad \log_{10}\frac{8x}{1+\sqrt{x}} = 2$; In North America, we write that $\log a = \log_{10} a$.

$\Rightarrow \qquad \frac{8x}{1+\sqrt{x}} = 10^2 = 100$

$\Rightarrow \qquad 8x = 100(1 + \sqrt{x})$

$\Rightarrow \qquad 2x = 25(1 + \sqrt{x}) = 25 + 25\sqrt{x}$; Divide both sides by 4.

$\Rightarrow \qquad 25\sqrt{x} = 2x - 25$

$\Rightarrow \qquad (25\sqrt{x})^2 = (2x - 25)^2$

$\Rightarrow \qquad 625x = 4x^2 - 100x + 625$

$\Rightarrow \qquad 4x^2 - 100x + 625 - 625x = 0$

$\Rightarrow \qquad 4x^2 - 725x + 625 = 0$; Here, $a = 4$, $b = -725$ and $c = 625$.

$\Rightarrow \qquad x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$\Rightarrow \qquad x = \frac{-(-725) \pm \sqrt{(-725)^2 - 4\times4\times625}}{2\times4}$

$$= \frac{725 \pm \sqrt{525625-10000}}{8} = \frac{725 \pm \sqrt{515625}}{8} = \frac{725 \pm \sqrt{5^6 \times 33}}{8} = \frac{725 \pm 5^3\sqrt{33}}{8} = \frac{725 \pm 125\sqrt{33}}{8}$$

$\Rightarrow \qquad x = \frac{725+125\sqrt{33}}{8}$ or $x = \frac{725-125\sqrt{33}}{8}$

Compound Interest Recall the following formulas:

- Interest compounded *n* times per year:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

- Interest compounded continuously

$$A(t) = Pe^{rt}$$

Examples:

[1] A woman invests $6,500 in an account that pays 6% interest per year, compounded continuously. How long will it take for the amount to be $8,000?

Solution: Formula $A(t) = Pe^{rt}$

Where we want $A(t) = \$8{,}000$
$P = \$6{,}500$
and $r = 6\% = 0.06$

We want to find *t*. So,

$$8000 = 6500e^{0.06t}$$

$\Rightarrow \qquad e^{0.06t} = \frac{8000}{6500} = \frac{16}{13}$

$\Rightarrow \qquad 0.06t = \ln\frac{16}{13}$

$\Rightarrow \qquad \frac{6}{100}t = \ln 16 - \ln 13$

$\Rightarrow \qquad t = \frac{100}{6}\ln\frac{16}{13}$

$\therefore \qquad t = \frac{50}{3}\ln\frac{16}{13}\ years$

[2] Find the time required for an investment of \$5,000 to grow to \$8,000 at an interest rate of 7.5% per year, compounded quarterly.

Solution: Formula $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$

Where we want $A(t) > \$8{,}000$
$P = \$5{,}000$
$r = 7.5\% = 0.075$
And $n = 4$, compounded quarterly or four times per year.

We want to find t. So,

$$5000\left(1 + \frac{0.075}{4}\right)^{4t} > 8000$$

$$\Rightarrow \quad \left(1 + \frac{0.075}{4}\right)^{4t} > \frac{8000}{5000}$$

$$\Rightarrow \quad \left(1 + \frac{0.075}{4}\right)^{4t} > 1.6$$

$$\Rightarrow \quad (1 + 0.01875)^{4t} > 1.6$$

$$\Rightarrow \quad 1.01875^{4t} > 1.6$$

$$\Rightarrow \quad \ln 1.01875^{4t} > \ln 1.6$$, taking the log of both sides.

$$\Rightarrow \quad 4t \ln 1.01875 > \ln 1.6$$

$$\Rightarrow \quad t > \frac{\ln 1.6}{4 \ln 1.01875}\ years$$

[3] How long will it take for an investment of \$1,000 to double in value if the interest rate is 8.5% per year, compounded continuously?

Solution: Formula $A(t) = Pe^{rt}$

We want $A(t) = \$2{,}000$, the investment is doubled.
$P = \$1{,}000$
And $r = 8.5\% = 0.085$

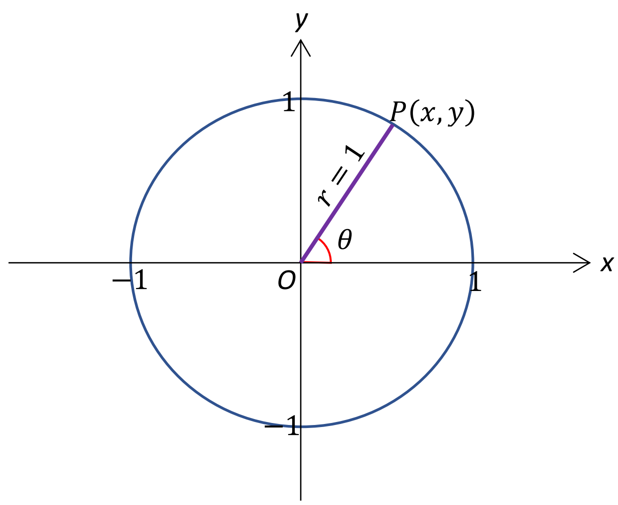
So, $2000 = 1000e^{0.085t}$

$$\Rightarrow \quad e^{0.085t} = \frac{2000}{1000} = 2$$

$$\Rightarrow \quad 0.085t = \ln 2 \qquad \therefore \qquad t = \frac{\ln 2}{0.085}\ years$$

5.1 Trigonometric Functions of Real Numbers

The Unit Circle is the circle of radius 1 unit with its centre at $O(0,0)$.

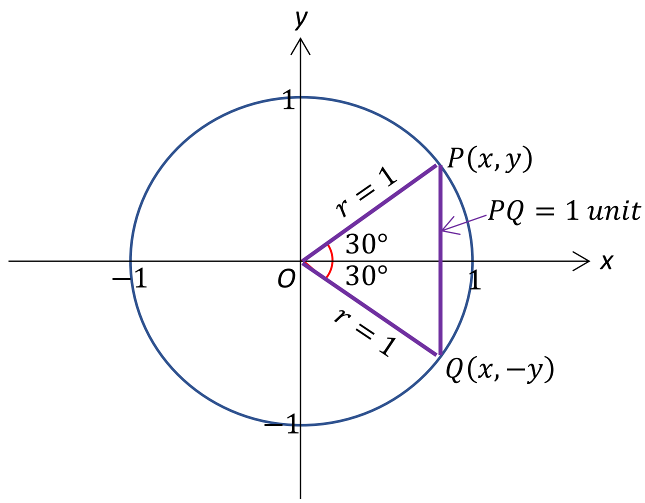


Let θ be the arc length from $O(0,0)$ to point $P(x,y)$. We can now define the trigonometric ratios as follow:

- $\sin\theta = y$
- $\cos\theta = x$
- $\tan\theta = \frac{y}{x}$, provided $x \neq 0$
- $\csc\theta = \frac{1}{y}$, provided $y \neq 0$
- $\sec\theta = \frac{1}{x}$, provided $x \neq 0$
- $\cot\theta = \frac{x}{y}$, provided $y \neq 0$

Based on the above definitions, we shall now define the trigonometric ratios for the angles $30°$, $45°$ and $60°$.

Trigonometric ratios for $30°$



Let us define an equilateral triangle *OPQ* placed on a Cartesian plane such that

- $OP = OQ = PQ = 1\,unit$ in length,
- Angle that *OP* makes with the *x*-axis is $30°$, and
- Angle that *OQ* makes with the *x*-axis is also $30°$.

We can now see from the above diagram that $\angle POQ = 60°$. It is obvious that

- Both points *P* and *Q* have the same *x*-coordinate and
- The *y*-coordinate of point *P* has the opposite sign of the *y*-coordinate of point *Q* although both have the same absolute value.

Using the distance formula, we obtain

$$OP = \sqrt{(x-0)^2 + (y-0)^2} = 1$$

$$\Rightarrow \qquad \sqrt{x^2 + y^2} = 1$$

$$\Rightarrow \qquad x^2 + y^2 = 1 \qquad\qquad (1)$$

And $\quad PQ = \sqrt{(x-x)^2 + \left(y-(-y)\right)^2} = 1$

$$\Rightarrow \qquad \sqrt{0^2 + (y+y)^2} = 1$$

$\Rightarrow \qquad \sqrt{(2y)^2} = 1$

$\Rightarrow \qquad 2y = 1$

$\Rightarrow \qquad y = \frac{1}{2}$

Substituting this into (1), we have

$x^2 + y^2 = 1$

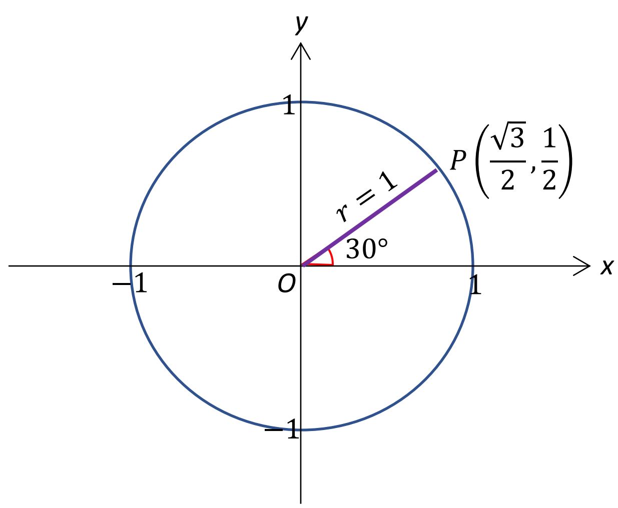
$\Rightarrow \qquad x^2 + \left(\frac{1}{2}\right)^2 = 1$

$\Rightarrow \qquad x^2 + \frac{1}{4} = 1$

$\Rightarrow \qquad x^2 = 1 - \frac{1}{4} = \frac{3}{4}$

$\Rightarrow \qquad x = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

And the above diagram simplifies to



For an angle of 30° defined from the *x*-axis to the terminal line, the terminal point is at $P\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

Therefore, the trigonometric ratios are

- $\sin 30° = y = \frac{1}{2}$
- $\cos 30° = x = \frac{\sqrt{3}}{2}$

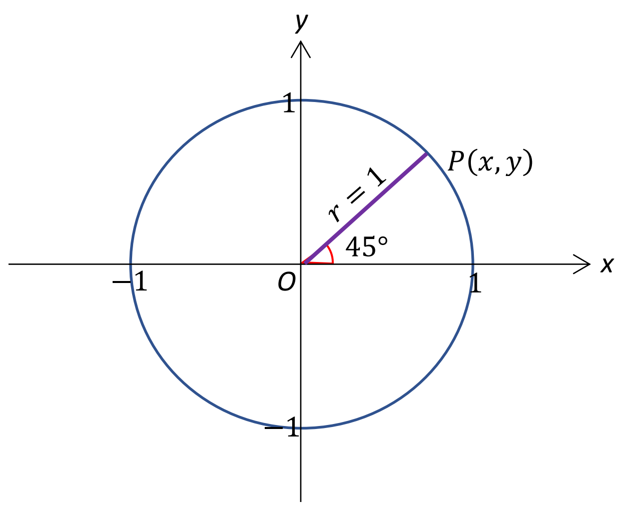
- $\tan 30° = \frac{y}{x} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

- $\csc 30° = \frac{1}{y} = \frac{1}{\frac{1}{2}} = 2$

- $\sec 30° = \frac{1}{x} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

- $\cot 30° = \frac{x}{y} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$

Trigonometric ratios for $45°$



When the terminal line defines an angle of $45°$ with the *x*-axis, its equation is $y = x$, i.e. the *x*-coordinate and the *y*-coordinate of point *P* have the same value. So we have,

$$\sqrt{(x-0)^2 + (y-0)^2} = 1$$

$$\Rightarrow \quad \sqrt{x^2 + y^2} = 1$$

$$\Rightarrow \quad x^2 + y^2 = 1 \qquad (1)$$

And we have $y = x$ that we substitute into (1).

$$x^2 + x^2 = 1$$

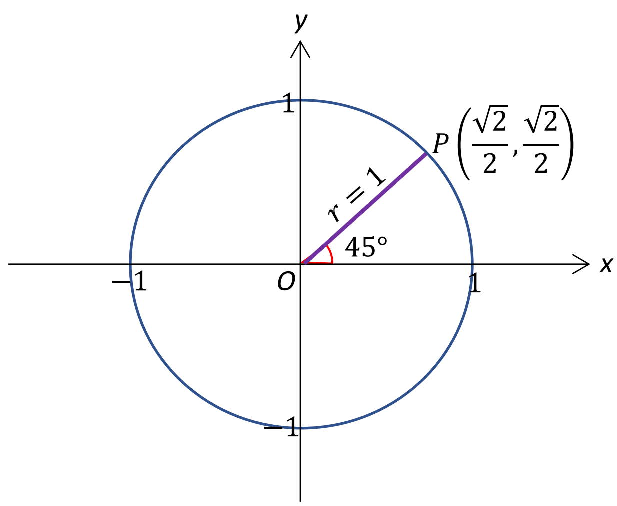
$$\Rightarrow \quad 2x^2 = 1$$

$$\Rightarrow \quad x^2 = \frac{1}{2}$$

$$\Rightarrow \qquad x = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

And also $\qquad y = \frac{\sqrt{2}}{2}$

And the above diagram simplifies to

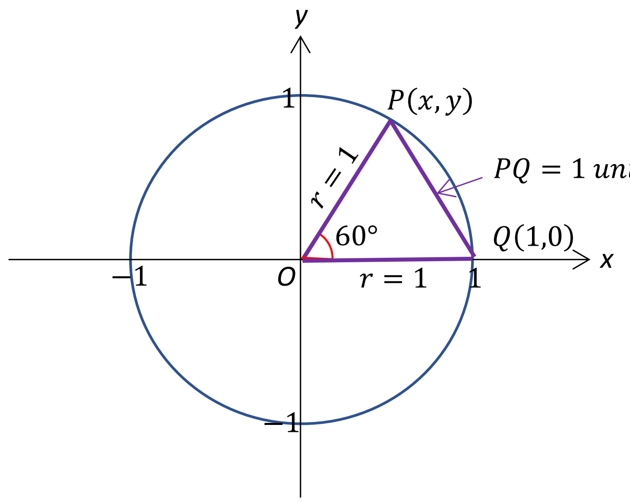


For an angle of 45° defined from the *x*-axis to the terminal line, the terminal point is at $P\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Therefore, the trigonometric ratios are

- $\sin 45° = y = \frac{\sqrt{2}}{2}$
- $\cos 45° = x = \frac{\sqrt{2}}{2}$
- $\tan 45° = \frac{y}{x} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$
- $\csc 45° = \frac{1}{y} = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$
- $\sec 45° = \frac{1}{x} = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$
- $\cot 45° = \frac{x}{y} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$

Trigonometric ratios for $60°$



Let us define an equilateral triangle *OPQ* placed on a Cartesian plane such that

- Point Q is located at coordinates $(1,0)$,
- $OP = OQ = PQ = 1\ unit$ in length, and
- $\angle POQ = \angle OPQ = \angle OQP = 60°$,

Using the distance formula, we obtain

$$OP = \sqrt{(x-0)^2 + (y-0)^2} = 1$$

$$\Rightarrow \quad \sqrt{x^2 + y^2} = 1$$

$$\Rightarrow \quad x^2 + y^2 = 1 \qquad (1)$$

And $\quad PQ = \sqrt{(x-1)^2 + (y-0)^2} = 1$

$$\Rightarrow \quad (x-1)^2 + (y-0)^2 = 1$$

$$\Rightarrow \quad x^2 - 2x + 1 + y^2 = 1$$

$$\Rightarrow \quad x^2 + y^2 - 2x = 0$$

$\Rightarrow \quad 1 - 2x = 0$, since $x^2 + y^2 = 1$ from equation (1)

$$\Rightarrow \quad 2x = 1$$

$$\Rightarrow \quad x = \frac{1}{2}$$

Substituting this result into (1),

$$x^2 + y^2 = 1$$

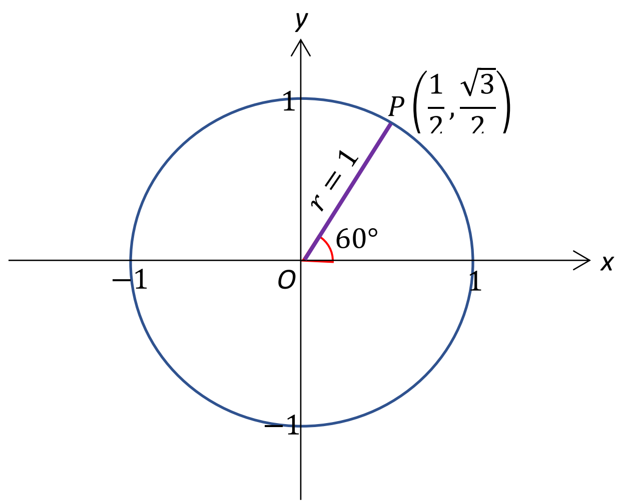
$$\Rightarrow \qquad \left(\frac{1}{2}\right)^2 + y^2 = 1$$

$$\Rightarrow \qquad \frac{1}{4} + y^2 = 1$$

$$\Rightarrow \qquad y^2 = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\Rightarrow \qquad y = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

And the above diagram simplifies to



For an angle of 60° defined from the *x*-axis to the terminal line, the terminal point is at $P\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

Therefore, the trigonometric ratios are

- $\sin 60° = y = \frac{\sqrt{3}}{2}$
- $\cos 60° = x = \frac{1}{2}$
- $\tan 60° = \frac{y}{x} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$
- $\csc 60° = \frac{1}{y} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

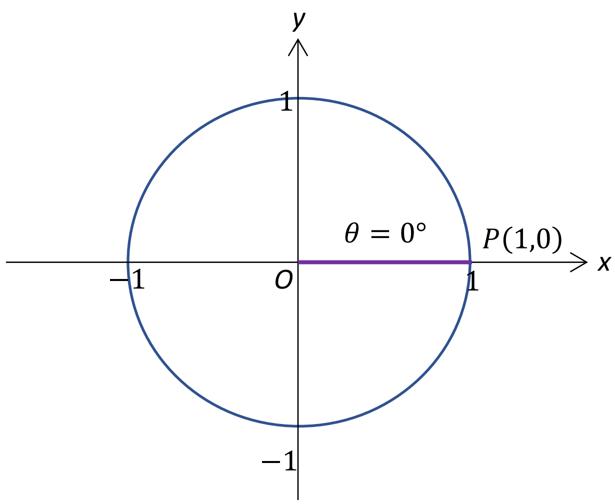
- $\sec 60° = \frac{1}{x} = \frac{1}{\frac{1}{2}} = 2$

- $\cot 60° = \frac{x}{y} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

Trigonometric Functions of Quadrant Angles

Now, we shall use the same Unit Circle to obtain the trigonometric functions of angles $0°$, $90°$, $180°$ and $270°$.

$\mathbf{0}°$ **:** The terminal point P lies on $(1,0)$

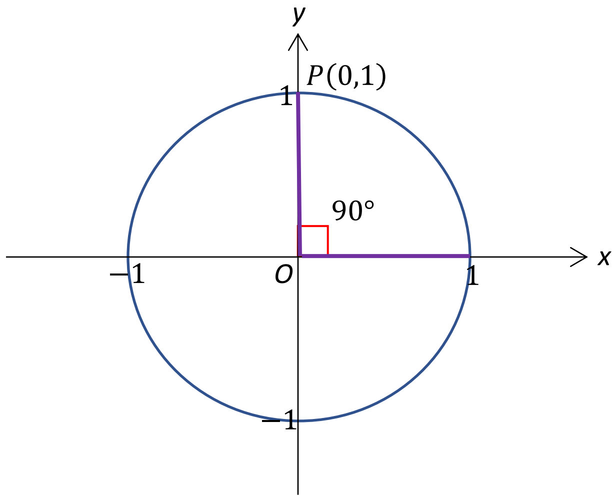


- $\sin 0° = y = 0$

- $\cos 0° = x = 1$

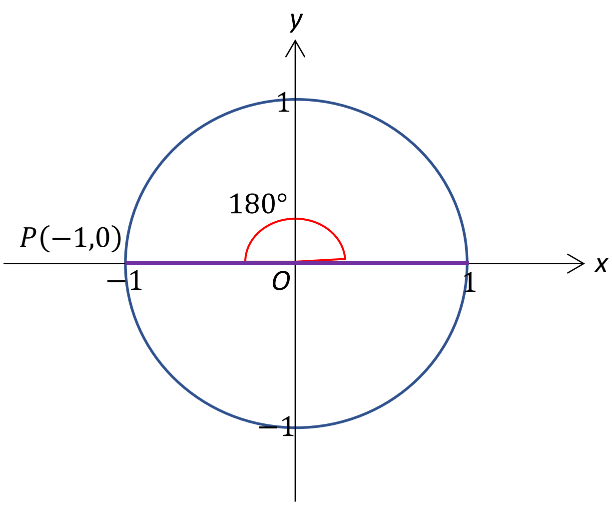
- $\tan 0° = \frac{y}{x} = \frac{0}{1} = 0$

$90°$: The terminal point *P* lies on $(0,1)$



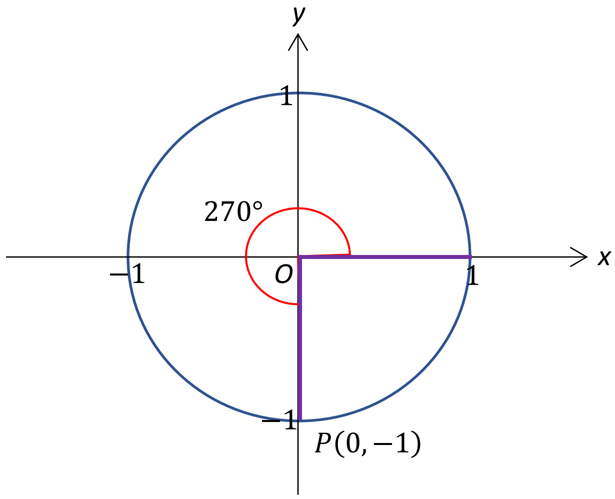
- $\sin 90° = y = 1$
- $\cos 90° = x = 0$
- $\tan 90° = \frac{y}{x} = \frac{1}{0}$ is undefined because we have a division by zero.

$180°$: The terminal point *P* lies on $(-1,0)$



- $\sin 180° = y = 0$
- $\cos 180° = x = -1$
- $\tan 180° = \frac{y}{x} = \frac{0}{-1} = 0$

$\mathbf{270°}$ **:** The terminal point *P* lies on $(0, -1)$



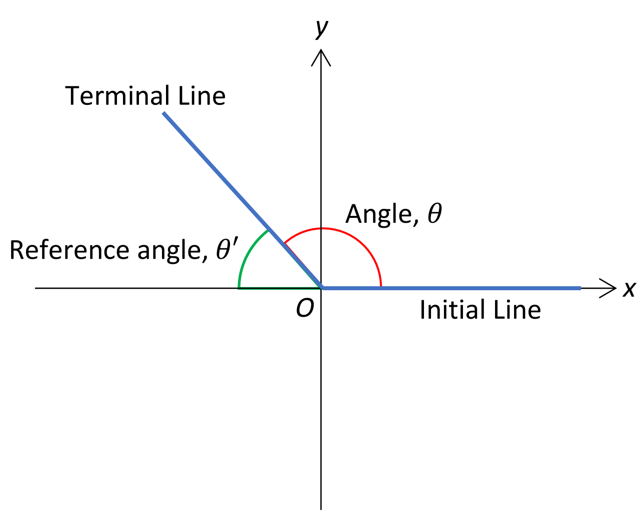
- $\sin 270° = y = -1$
- $\cos 270° = x = 0$
- $\tan 270° = \frac{y}{x} = \frac{-1}{0}$ is undefined because we have a division by zero.

Reference Angle

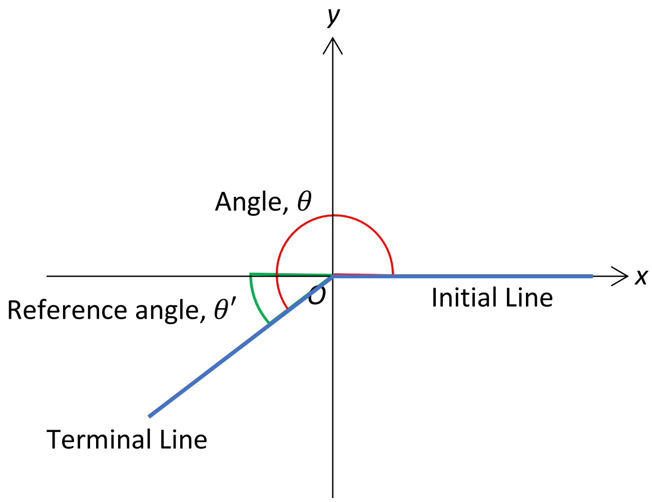
Recall that we define an angle θ from the positive side of the *x*-axis up to the terminal line. Its reference angle θ' is the acute angle from any part of the *x*-axis and in any direction to this terminal line.

Diagrams:

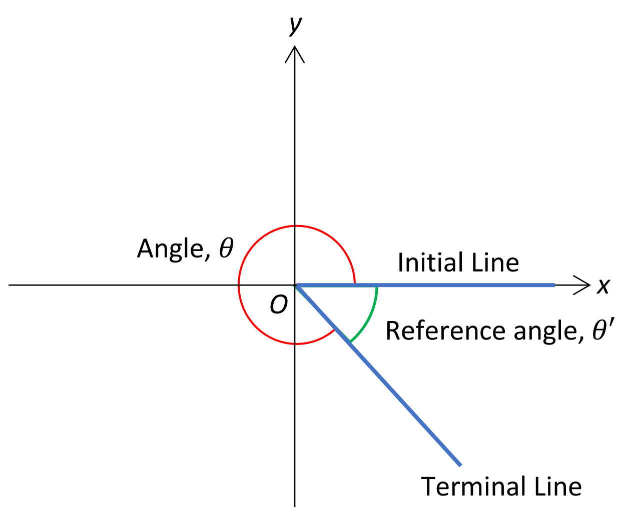
1. If $90° < \theta < 180°$



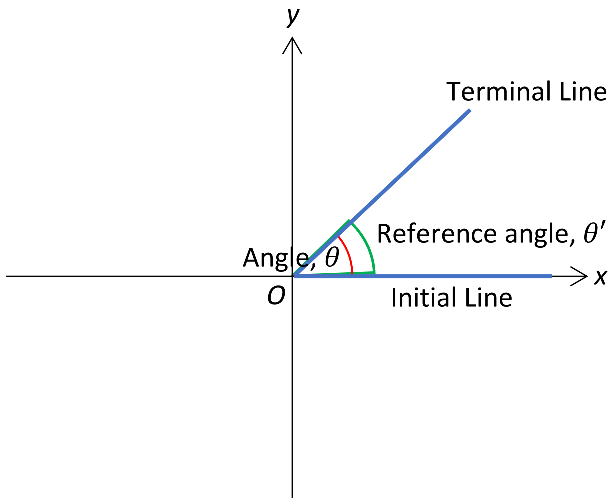
2. If $180° < \theta < 270°$



3. If $270° < \theta < 360°$

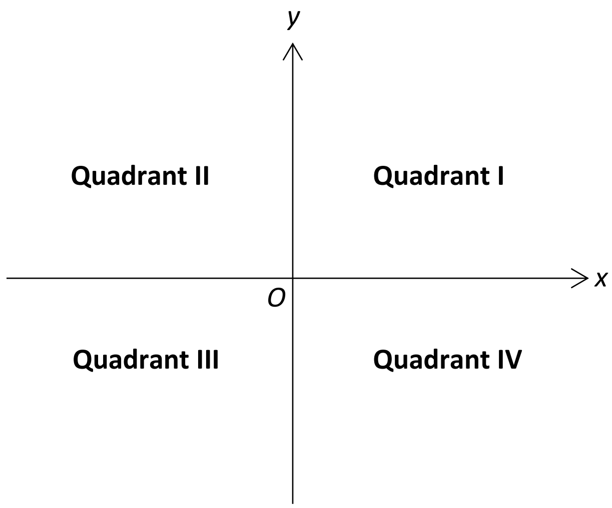


4. And of course if $0° < \theta < 90°$



Quadrants

The Cartesian plane is divided into four quadrants by the *x* and *y* axes.

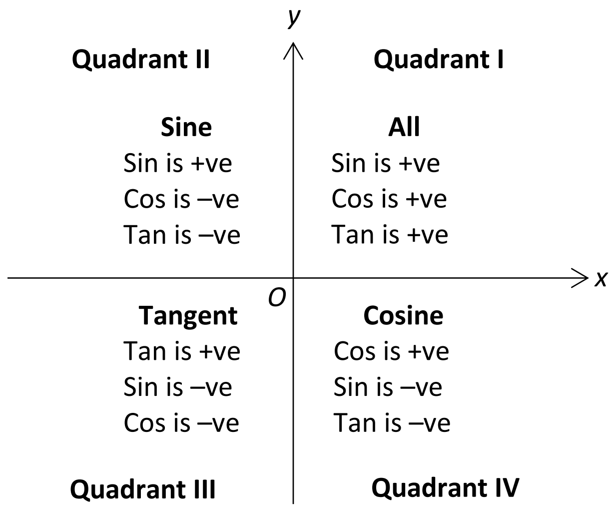


Evaluating Trigonometric Functions of Any Angle

To find the trigonometric function of any angle θ:

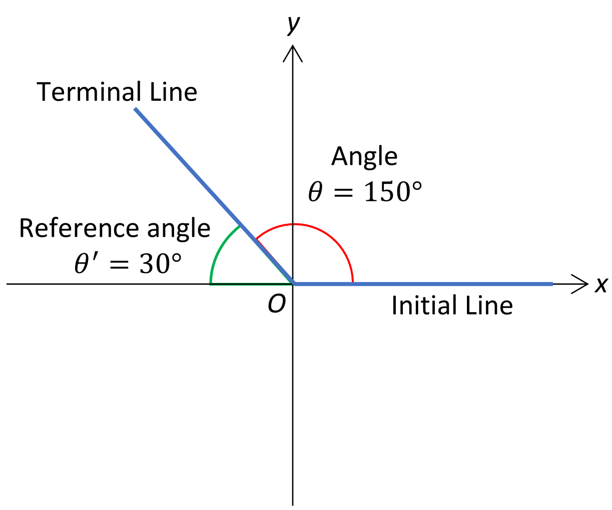
1. Determine the function value for the corresponding reference angle θ'.

2. Assign the appropriate sign to the trigonometric function depending on the quadrant the terminal line falls in. To obtain the appropriate sign, we use the "CAST" Rule:



Examples: Find the trigonometric functions of the following angles:

1. $150°$



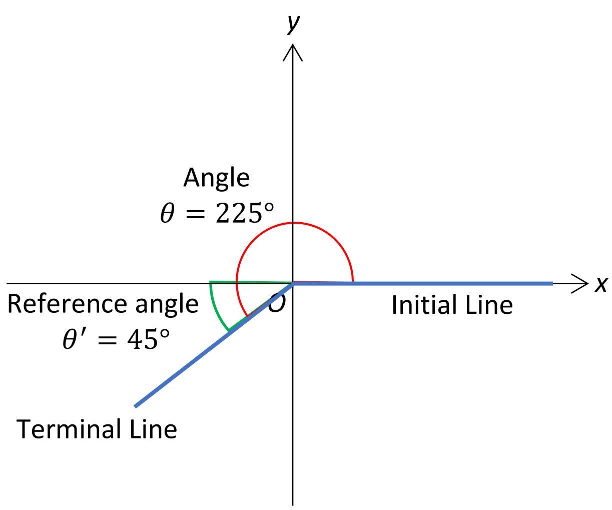
Since the terminal line falls in the Second Quadrant, only the Sine function is positive while the Cosine and the Tangent functions are negative. So we have

$$\sin 150° = \sin 30° = \frac{1}{2}$$

$$\cos 150° = -\cos 30° = -\frac{\sqrt{3}}{2}$$

And $\quad \tan 150° = -\tan 30° = -\frac{\sqrt{3}}{3}$

2. $\quad$ 225°



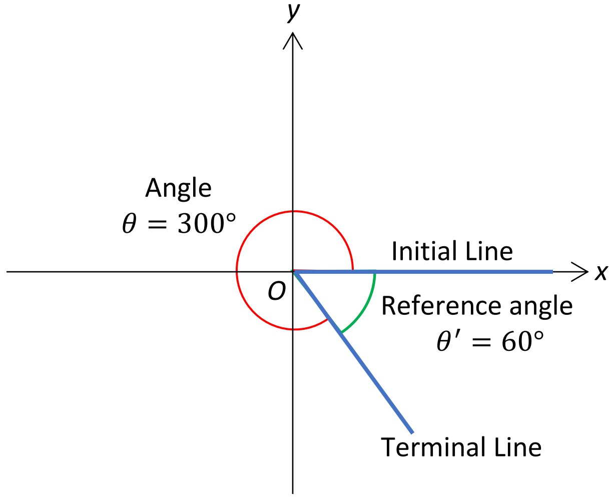
Since the terminal line falls in the Third Quadrant, only the Tangent function is positive while the Sine and the Cosine functions are negative. So we have

$$\sin 225° = -\sin 45° = -\frac{\sqrt{2}}{2}$$

$$\cos 225° = -\cos 45° = -\frac{\sqrt{2}}{2}$$

And $\quad \tan 225° = \tan 45° = 1$

3. 300°



Since the terminal line falls in the Fourth Quadrant, only the Cosine function is positive while the Sine and the Tangent functions are negative. So we have

$$\sin 300° = -\sin 60° = -\frac{\sqrt{3}}{2}$$

$$\cos 300° = \cos 60° = \frac{1}{2}$$

And $\tan 300° = -\tan 60° = -\sqrt{3}$

Summary

Up to this stage we have already collected the exact trigonometric functions of 0°, 30°, 45°, 60°, 90°, 180° and 270° as shown in the table below:

$\boldsymbol{\theta}$ **(degrees)**	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$
$\boldsymbol{\theta}$ **(radians)**	$0\ rad$	$\frac{\pi}{6}\ rad$	$\frac{\pi}{4}\ rad$	$\frac{\pi}{3}\ rad$	$\frac{\pi}{2}\ rad$	$\pi\ rad$	$\frac{3\pi}{2}\ rad$
$\boldsymbol{\sin\theta}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\boldsymbol{\cos\theta}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\boldsymbol{\tan\theta}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined	0	Undefined

Finally, we end this section by summarising as follow:

Reciprocal Trigonometric Functions

- $\csc\theta = \frac{1}{\sin\theta}$
- $\sec\theta = \frac{1}{\cos\theta}$
- $\cot\theta = \frac{1}{\tan\theta}$

Even-Odd Properties

- $\sin(-x) = -\sin x$; $\sin x$ is an odd function.
- $\cos(-x) = \cos x$; $\cos x$ is an even function.
- $\tan(-x) = -\tan x$; $\tan x$ is an odd function.
- $\csc(-x) = -\csc x$; $\csc x$ is an odd function.
- $\sec(-x) = \sec x$; $\sec x$ is an even function.
- $\cot(-x) = -\cot x$; $\cot x$ is an odd function.

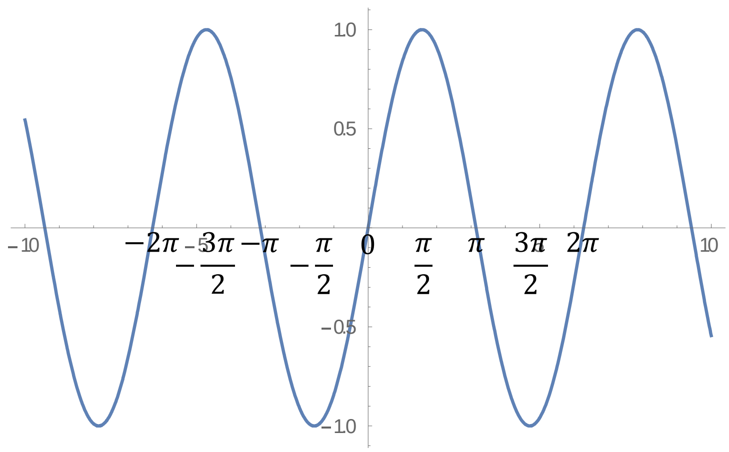
Pythagorean Identities

- $\sin^2 x + \cos^2 x = 1$
- $1 + \tan^2 x = \sec^2 x$
- $1 + \cot^2 x = \csc^2 x$

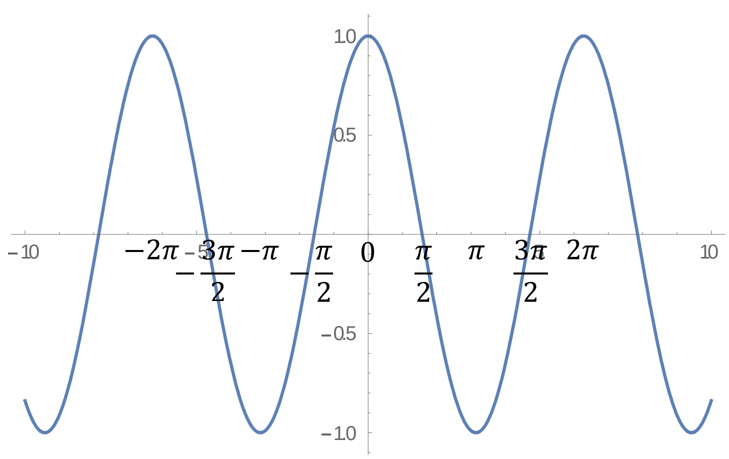
5.2 Graphs of Trigonometric Functions

You should memorise the following trigonometric graphs:

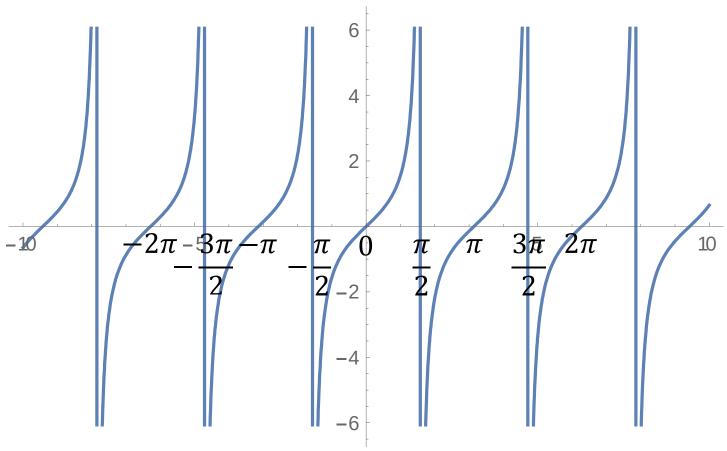
Graph of $y = \sin x$:



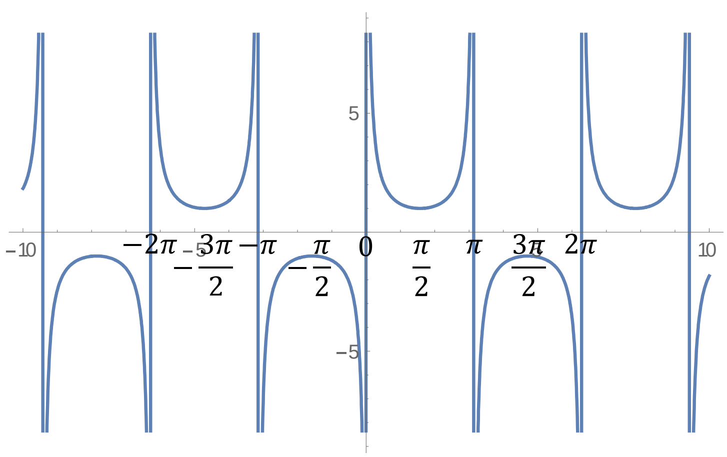
Graph of $y = \cos x$:



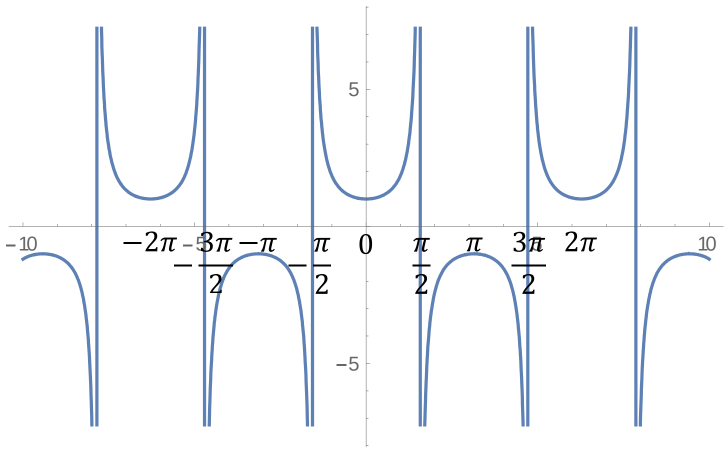
Graph of $y = \tan x$:



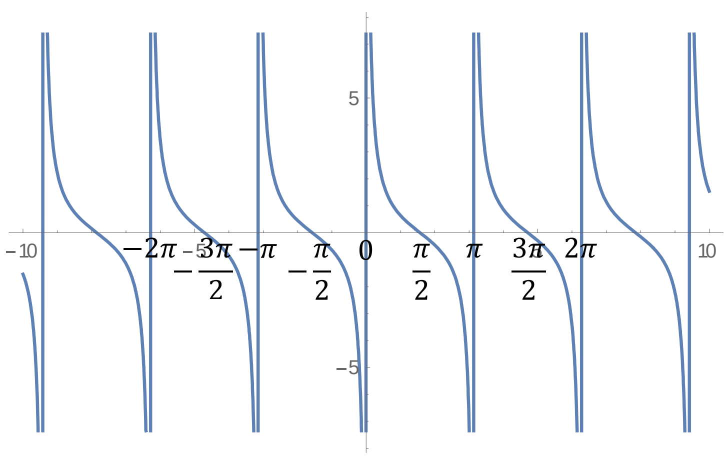
Graph of $y = \csc x$:



Graph of $y = \sec x$:



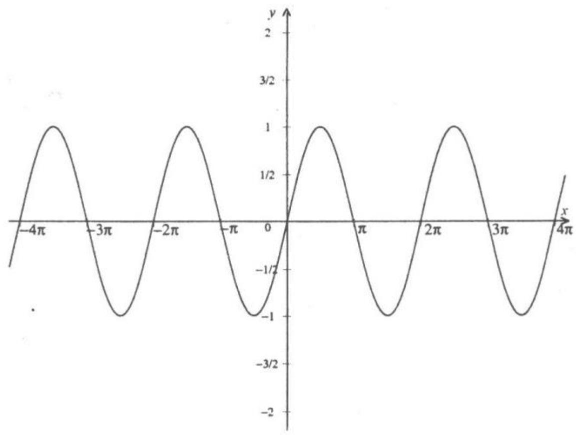
Graph of $y = \cot x$:



5.3 Inverse Trigonometric Functions and Their Graphs

Inverse Sine Function

The original sine function, $y = \sin x$, has a domain that is $x \in (-\infty, \infty)$ and as you can see from the graph of $y = \sin x$, this function is not one-to-one. However, if we restrict the domain to an interval $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, the function does become one-to-one.

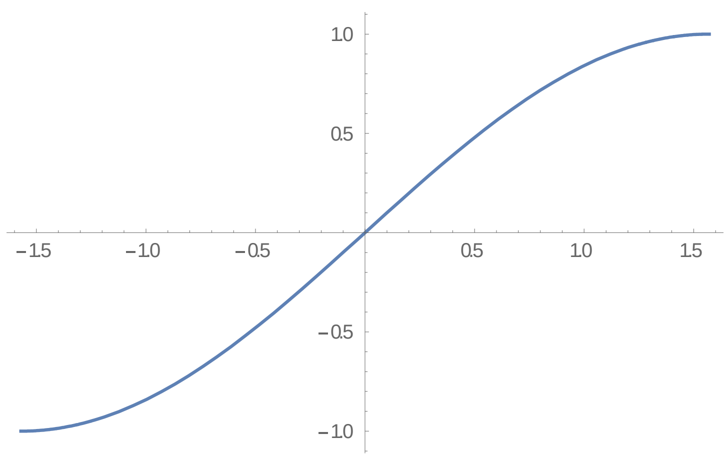


Graph of $y = \sin x$

The reason why we restrict the domain to an interval where the function $y = \sin x$ becomes one-to-one is so that we can define its inverse function. Hence, if

$y = \sin x$ defined on the interval $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ with range $-1 \le y \le 1$

Graph of $y = \sin x$ defined on $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$:



Its inverse function is defined as

$$x = \sin^{-1} y \qquad \text{or} \qquad x = \arcsin y$$

However, as you may recall, we usually use the symbol *x* for the independent variable and the symbol *y* for the dependent variable. Therefore, we have the definition of the inverse sine function as:

The inverse sine function is defined by

$$y = \sin^{-1} x = \arcsin x \qquad \Leftrightarrow \qquad \sin y = x$$

Where the domain is $-1 \le x \le 1$ or $x \in [-1,1]$

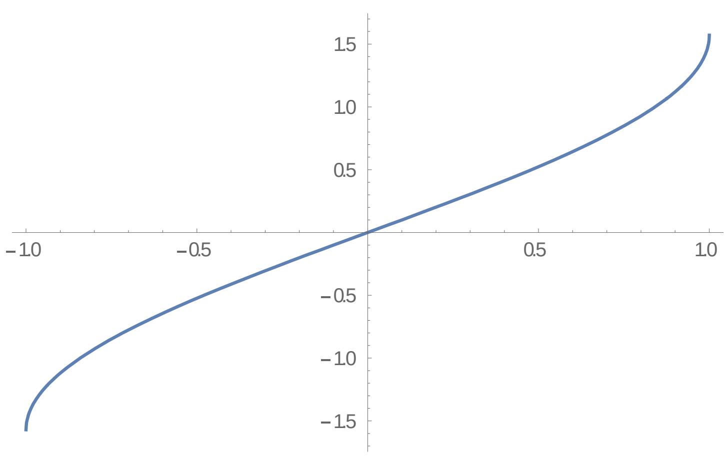
And the range is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ or $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Similarly, we can define the inverse of the other trigonometric functions with their domains and ranges:

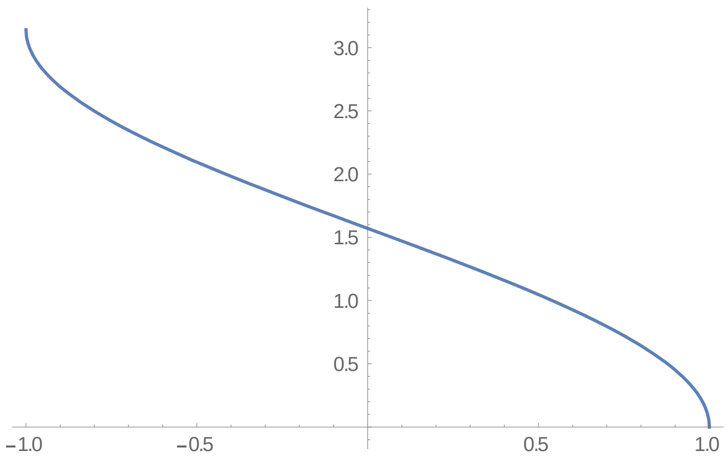
Definitions of the Inverse Trigonometric Functions

Trigonometric Function	**Inverse Trigonometric Function**	**Domain**	**Range**
$\sin y = x$	$y = \sin^{-1} x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$\cos y = x$	$y = \cos^{-1} x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$\tan y = x$	$y = \tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

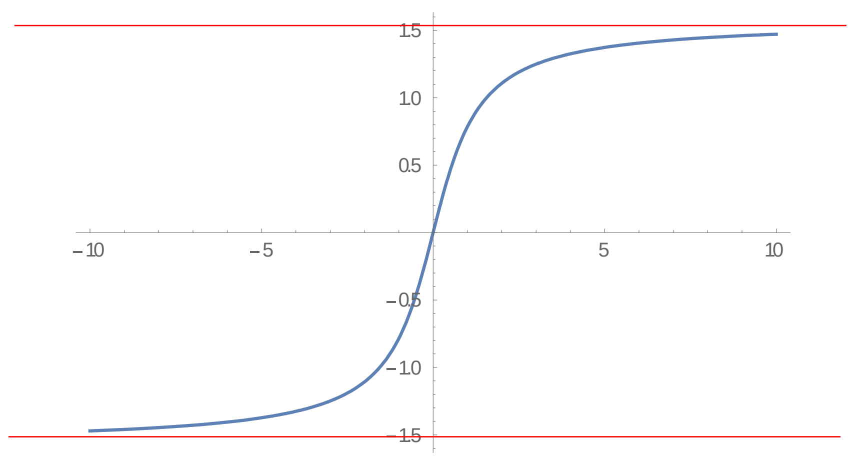
Graph of $y = \sin^{-1} x$:



Graph of $= \cos^{-1} x$:



Graph of $= \tan^{-1} x$:



Examples: Evaluate the following:

1. $\sin^{-1}\left(-\frac{1}{2}\right)$

Solution: Let $y = \sin^{-1}\left(-\frac{1}{2}\right)$

$\Leftrightarrow \quad \sin y = -\frac{1}{2}$

Now, we know that for the $y = \sin^{-1} x$ function, the range is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. Therefore, when $\sin y = -\frac{1}{2}$, the only result we can obtain is $y = -\frac{\pi}{6}$. Thus,

$\therefore \quad \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

2. $\cos^{-1}\frac{\sqrt{2}}{2}$

Solution: Let $y = \cos^{-1}\frac{\sqrt{2}}{2}$

$\Leftrightarrow \quad \cos y = \frac{\sqrt{2}}{2}$

Now, we know that for the $y = \cos^{-1} x$ function, the range is $0 \leq y \leq \pi$. Therefore, when $\cos y = \frac{\sqrt{2}}{2}$, the only result we can obtain is $y = \frac{\pi}{4}$. Thus,

$\therefore \quad \cos^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$

3. $\tan^{-1}(-1)$

Solution: Let $y = \tan^{-1}(-1)$

$\Leftrightarrow \quad \tan y = -1$

Now, we know that for the $y = \tan^{-1} x$ function, the range is $-\frac{\pi}{2} < y < \frac{\pi}{2}$. Therefore, when $\tan y = -1$, the only result we can obtain is $y = -\frac{\pi}{4}$. Thus,

$\therefore \quad \tan^{-1}(-1) = -\frac{\pi}{4}$

4. $\tan\left(\cos^{-1}\frac{2}{3}\right)$

Solution: Let $u = \cos^{-1}\frac{2}{3}$

$\Leftrightarrow \quad \cos u = \frac{2}{3}$

i. Since $\cos u = \frac{2}{3} > 0$, we know that *u* must be an angle terminating in either the first quadrant or the fourth quadrant based on the "CAST" rule.

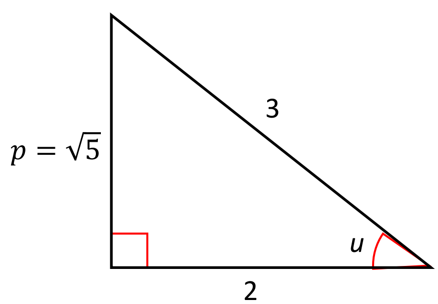
ii. The range for $y = \cos^{-1} x$ is

$$0 \leq \cos^{-1} x \leq \pi$$

Which is in either the first quadrant or the second quadrant,

Conclusion: By taking the overlapping of both results above, we conclude that *u* must be an angle in the first quadrant.

Drawing a right triangle with angle *u*, and knowing that $\cos u = \frac{2}{3}$, we have



Let *p* be the length of the opposite side. Using the Pythagorean Theorem, we can calculate that

$$p^2 + 2^2 = 3^2$$

$$\Rightarrow \quad p^2 + 4 = 9$$

$$\Rightarrow \quad p^2 = 9 - 4 = 5$$

$$\Rightarrow \quad p = \sqrt{5}$$

Therefore,

$$\tan\left(\cos^{-1}\frac{2}{3}\right) = \tan u = \frac{\sqrt{5}}{2}$$

Taking a positive value since the angle *u* terminates in the first quadrant.

5. $\cos\left(\sin^{-1}\left(-\frac{3}{5}\right)\right)$

Solution: Let $u = \sin^{-1}\left(-\frac{3}{5}\right)$

$\Leftrightarrow \quad \sin u = -\frac{3}{5}$

i. Since $\sin u = -\frac{3}{5} < 0$, we know that *u* must be an angle terminating in either the third quadrant or the fourth quadrant based on the "CAST" rule.

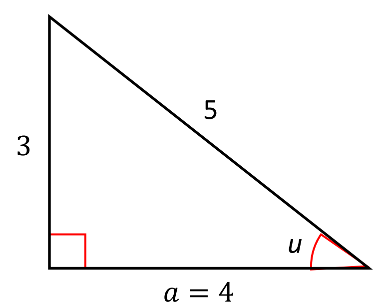
ii. The range for $y = \sin^{-1} x$ is

$$-\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2}$$

Which is in either the first quadrant or the fourth quadrant,

Conclusion: By taking the overlapping of both results above, we conclude that *u* must be an angle in the fourth quadrant.

Drawing a right triangle with angle *u*, and knowing that $\sin u = -\frac{3}{5}$, we have by taking only positive lengths for the sides



Let *a* be the length of the adjacent side. Using the Pythagorean Theorem, we can calculate that

$3^2 + a^2 = 5^2$

$\Rightarrow \quad a^2 + 9 = 25$

$\Rightarrow \quad a^2 = 25 - 9 = 16$

$\Rightarrow \quad a = \sqrt{16} = 4$

Therefore,

$$\cos\left(\sin^{-1}\left(-\frac{3}{5}\right)\right) = \cos u = \frac{4}{5}$$

Taking a positive value since the angle *u* terminates in the fourth quadrant where the cosine function takes a positive value.

6. $\sin(\cos^{-1}(3x))$, where $0 \leq x \leq \frac{1}{3}$

Solution: Let $u = \cos^{-1}(3x)$

$\Rightarrow$ $\cos u = 3x$

We are given that $0 \leq x \leq \frac{1}{3}$

$\Rightarrow$ $0 \leq 3x \leq 1$

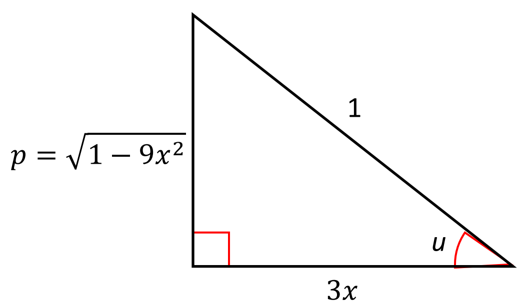
Meaning that $\cos u = 3x$ is a non-negative value.

i. Hence, we know that *u* is an angle terminating in either the first or the fourth quadrant from the "CAST" rule.

ii. Also, since the range of the arccosine function is $0 \leq \cos^{-1}(3x) \leq \pi$, the range falls either in the first quadrant or the second quadrant.

Conclusion: Taking the overlapping of quadrants from the above two results, we conclude that *u* must terminate in the first quadrant.

Drawing a right triangle with angle *u*, and knowing that $\cos u = 3x = \frac{3x}{1}$, we have



Let *p* be the length of the opposite side. Using the Pythagorean Theorem, we can calculate that

$$p^2 + (3x)^2 = 1^2$$

$$\Rightarrow \quad p^2 + 9x^2 = 1$$

$$\Rightarrow \quad p^2 = 1 - 9x^2$$

$$\Rightarrow \quad p = \sqrt{1 - 9x^2}$$

Therefore,

$$\sin(\cos^{-1}(3x)) = \sqrt{1 - 9x^2} \ , \ \text{ with } 0 \leq x \leq \frac{1}{3}$$

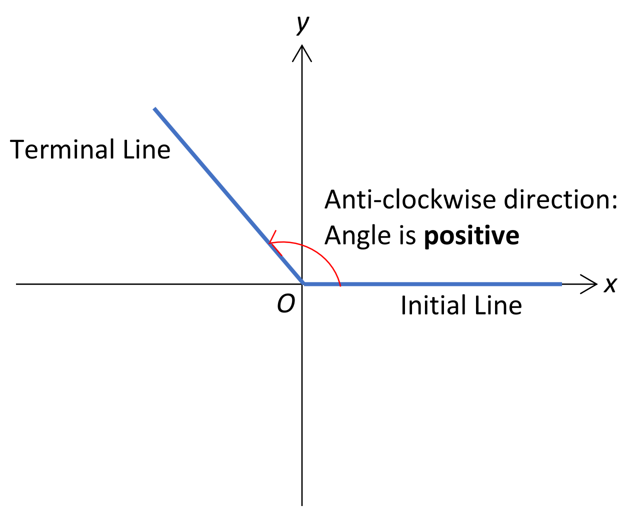
Taking a positive value since the angle *u* terminates in the first quadrant.

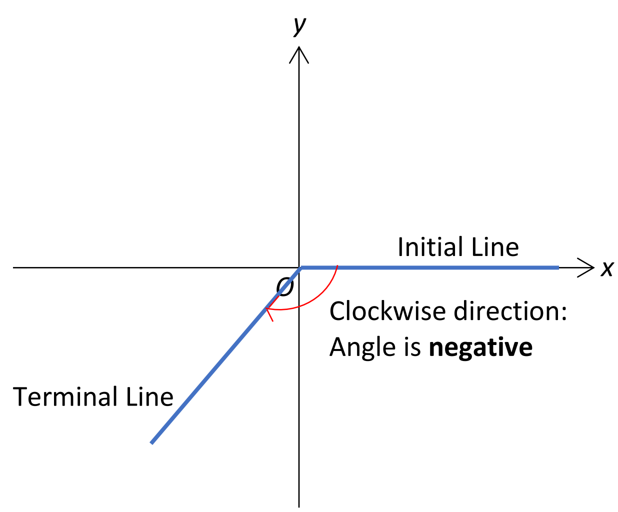
6.1 Angle Measure

Radian and Degree Measure

Some protocol when defining angles on a Cartesian coordinate system:

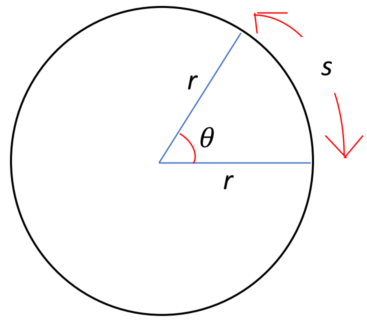
1. When we measure an angle, we always start from the **positive side** of the *x*-axis.
2. An angle that is measured **anti-clockwise** is given a **positive** value.
3. An angle that is measure **clockwise** is given a **negative** value.
4. The line that we draw after we have stopped measuring our angle is called the **terminal** line (side).





Radian

Let us consider a circle with radius, *r*, and centre, *C*.



Let us also consider an arc on the circle that defines an arc length *s*. This arc length defines an angle θ that is measured in **radians** as

$$\theta = \frac{s}{r}$$

Conversions between Radians and Degrees

The following rules will help:

1. $0\, rad = 0°$

And 2. $\pi\, rad = 180°$

Examples:

Convert the following angles from radians to degrees:

1. $\frac{\pi}{2}\, rad = \frac{1}{2} \times \pi\, rad$

$$= \frac{1}{2} \times 180° = 90°$$

2. $2\, rad = \frac{2}{\pi} \times \pi\, rad$

$$= \frac{2}{\pi} \times 180° = \left(\frac{360}{\pi}\right)^{o}$$

Convert the following angles from degrees to radians:

1. $30° = \frac{30°}{180°} \times 180°$

$= \frac{30°}{180°} \times \pi\, rad = \frac{1}{6}\pi\, rad$

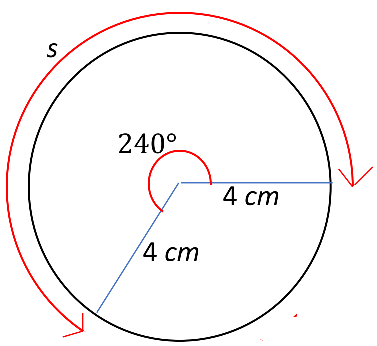
2. $45° = \frac{45°}{180°} \times 180°$

$= \frac{45°}{180°} \times \pi\, rad = \frac{1}{4}\pi\, rad$

3. $60° = \frac{60°}{180°} \times 180°$

$= \frac{60°}{180°} \times \pi\, rad = \frac{1}{3}\pi\, rad$

4. A circle has a radius of 4 *cm*. Find the length of the arc, *s*, defined by a central angle of 240°.



Solution: From the definition of an angle in radians:

$$\theta = \frac{s}{r}$$

$\Rightarrow$ $s = r\theta$, where θ is the central angle in radians.

Therefore, we must convert the given angle from degrees to radians:

$240° = \frac{240°}{180°} \times 180°$

$= \frac{240°}{180°} \times \pi\, rad = \frac{4}{3}\pi\, rad$

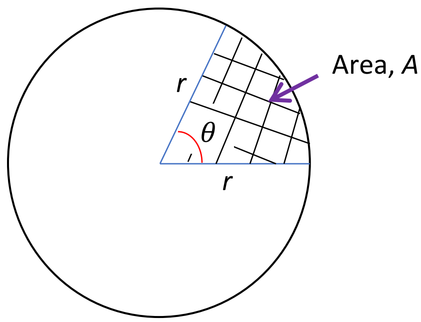
Now, given that the radius is 4 *cm*, we can calculate the arc length

$$s = r\theta$$

$$\Rightarrow \qquad s = 4 \times \frac{4}{3}\pi = \frac{16}{3}\pi\ cm$$

Area of a Sector of a Circle

Let us consider a circle with radius *r* as shown in the diagram below. Let a sector of the circle be defined by the central angle θ.



We know that the area of a complete circle is πr^2 and that the angle of a full circle is $360°$. Therefore, the area of the sector of a circle defined by the central angle θ is

$$A = \frac{\theta}{360°} \times \pi r^2$$

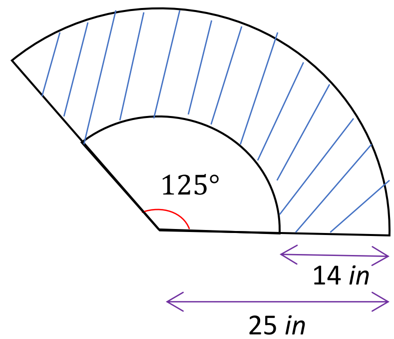
And $\qquad 360° = \frac{360°}{180°} \times 180°$

$$= \frac{360°}{180°} \times \pi\ rad = 2\pi\ rad$$

Therefore, $\qquad A = \frac{\theta}{2\pi} \times \pi r^2 = \frac{\theta}{2} r^2$

$\Rightarrow \qquad A = \frac{1}{2} r^2 \theta$, where θ is the central angle in radians.

Example: A car's rear windshield wiper rotates 125°. The total length of the wiper mechanism is 25 *in* and wipes the windshield over a distance of 14 *in*. Find the area covered by the wiper.



Solution:

Notice there are two sectors of circles involved. The bigger circle has radius

$$R = 25\ in$$

and the smaller circle has radius

$$r = 25 - 14 = 11\ in$$

Angle the wiper sweeps in radians:

$$\theta = 125° = \frac{125°}{180°} \times 180°$$

$$= \frac{125°}{180°} \times \pi\ rad = \frac{25}{36}\pi\ rad$$

Area of big sector:

$$A_{big} = \frac{1}{2} \times 25^2 \times \frac{25}{36}\pi = \frac{625}{2} \times \frac{25}{36}\pi\ in^2$$

Area of small sector:

$$A_{small} = \frac{1}{2} \times 11^2 \times \frac{25}{36}\pi = \frac{121}{2} \times \frac{25}{36}\pi\ in^2$$

Area that the wiper sweeps:

$$A_{sweep} = A_{big} - A_{small} = \frac{625}{2} \times \frac{25}{36}\pi - \frac{121}{2} \times \frac{25}{36}\pi$$

$$= \left(\frac{625}{2} - \frac{121}{2}\right) \times \frac{25}{36}\pi$$

$$= \frac{504}{2} \times \frac{25}{36}\pi$$

$$= 252 \times \frac{25}{36}\pi = 7 \times 25\pi = 175\pi\ in^2$$

Linear and Angular Speed

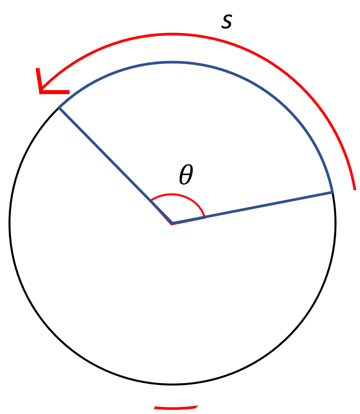
Let us consider a particle moving at a constant speed along a circular arc of radius *r*.

If *s* is the length of the arc traveled in time *t*, then the **Linear Speed** of the particle is

$$v = \frac{Change\ in\ Arc\ Length, s}{Time\ taken, t}$$

If θ is the central angle within that circle measured in radians, then the rate of change of the angle is called the **angular speed**, denoted by ω, i.e.

Angular speed, $\omega = \frac{Change\ in\ angle, \theta}{Time\ taken, t}$



Example:

The radius of the driving wheel of a bicycle is 14 *in*. A cyclist is pedaling and the driving wheel turns at the rate of 1 revolution every 2 seconds.

1. What is the angular speed of the wheel?

Answer: One complete revolution is $360° = 2\pi\ rad$

i.e. The wheel is turning with an angular speed of $\omega = \frac{2\pi}{2} = \pi\ rad\ per\ second$

2. What is the speed of the bicycle?

Answer: The speed of the bicycle is equal to the linear speed of a point on the wheel circle

i.e. The circumference of the wheel is $C = 2\pi r$

$$\Rightarrow \quad C = 2\pi \times 14 = 28\pi$$

The wheel turns at the rate of 1 revolution every 2 seconds, therefore the speed of the bicycle is

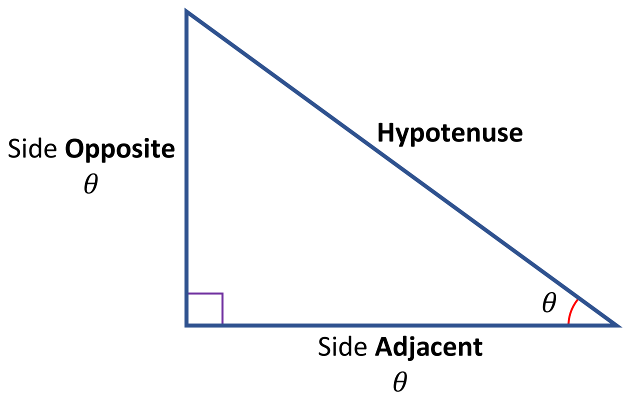
$$v = \frac{28\pi}{2} = 14\pi \; in \; per \; second$$

6.2 Trigonometry of Right Triangles

Trigonometric Ratios

This is the second method used to define trigonometric ratios.

Let us consider a right triangle with one of the vertices defining an angle θ that we are looking to find the trigonometric ratios of.



The trigonometric functions of the angle θ are defined as follow:

- $\sin\theta = \frac{Opposite}{Hypotenuse}$
- $\cos\theta = \frac{Adjacent}{Hypotenuse}$
- $\tan\theta = \frac{Opposite}{Adjacent}$
- $\csc\theta = \frac{Hypotenuse}{Opposite}$
- $\sec\theta = \frac{Hypotenuse}{Adjacent}$
- $\cot\theta = \frac{Adjacent}{Opposite}$

Evaluating the trigonometric functions of $30°$ and $60°$

We will use an equilateral triangle with sides of length 2 units that is divided into two equal right triangles by a perpendicular bisector.

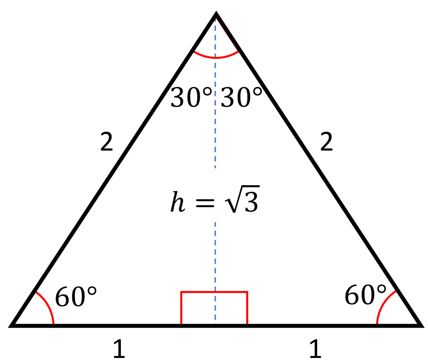
Let *h* be the height of the perpendicular bisector. Using the Pythagorean Theorem, we can calculate *h*:

$$h^2 + 1^2 = 2^2$$

$$\Rightarrow \qquad h^2 + 1 = 4$$

$$\Rightarrow \qquad h^2 = 4 - 1 = 3$$

$$\Rightarrow \qquad h = \sqrt{3}$$



By observing any one of the right triangles, we can define the following ratios:

$30°$:

- $\sin 30° = \frac{1}{2}$
- $\cos 30° = \frac{\sqrt{3}}{2}$
- $\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
- $\csc 30° = 2$
- $\sec 30° = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$
- $\cot 30° = \sqrt{3}$

$60°$:

- $\sin 60° = \frac{\sqrt{3}}{2}$
- $\cos 60° = \frac{1}{2}$
- $\tan 60° = \sqrt{3}$
- $\csc 60° = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$
- $\sec 60° = 2$
- $\cot 60° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

Evaluating the trigonometric functions of $45°$

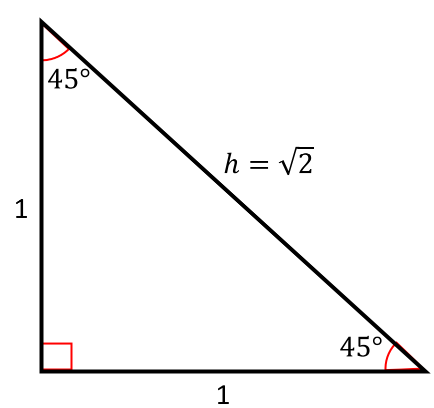
We will use a right isosceles triangle with the two equal length sides having length 1 unit.

Let *h* be the length of the hypotenuse. Using the Pythagorean Theorem, we can find *h*.

$$1^2 + 1^2 = h^2$$

$$\Rightarrow \quad h^2 = 1 + 1 = 2$$

$$\Rightarrow \quad h = \sqrt{2}$$



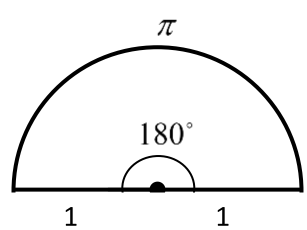
We can now define the trigonometric functions of $45°$:

- $\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
- $\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
- $\tan 45° = 1$
- $\csc 45° = \sqrt{2}$
- $\sec 45° = \sqrt{2}$
- $\cot 45° = 1$

Summary

How do we convert an angle from degrees to radians and vice versa? Simple, just remember that

$$\pi \text{ rad} = 180^\circ$$



Examples

[1] $36^\circ = \frac{36}{180} \times 180^\circ = \frac{36}{180}\pi\, rad = \frac{1}{5}\pi\, rad$

[2] $-\frac{7\pi}{2}\, rad = -\frac{7}{2} \times \pi\, rad = -\frac{7}{2} \times 180^\circ = -630^\circ$

[3] $5\, rad = \frac{5}{\pi}\pi\, rad = \frac{5}{\pi} \times 180^\circ = \frac{900^\circ}{\pi}$

Up to this stage we have already collected the exact trigonometric functions of 0°, 30°, 45°, 60°, 90°, 180° and 270° as shown in the table below:

$\boldsymbol{\theta}$ **(degrees)**	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$
$\boldsymbol{\theta}$ **(radians)**	$0\, rad$	$\frac{\pi}{6}\, rad$	$\frac{\pi}{4}\, rad$	$\frac{\pi}{3}\, rad$	$\frac{\pi}{2}\, rad$	$\pi\, rad$	$\frac{3\pi}{2}\, rad$
$\boldsymbol{\sin\theta}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\boldsymbol{\cos\theta}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\boldsymbol{\tan\theta}$	0	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined	0	Undefined

Applications of Trigonometry of Right Triangles

1. Let θ be an acute angle such that $\sin\theta = 0.6$. Find the values of

(a) $\cos\theta$

Solution: From the identity $\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow$ $0.6^2 + \cos^2\theta = 1$

$\Rightarrow$ $0.36 + \cos^2\theta = 1$

$\Rightarrow$ $\cos^2\theta = 1 - 0.36 = 0.64$

$\Rightarrow$ $\cos\theta = \sqrt{0.64} = 0.8$

(b) $\tan\theta$

Solution: From the identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\Rightarrow$ $\tan\theta = \frac{0.6}{0.8} = \frac{3}{4}$

2. Let θ be an acute angle such that $\tan\theta = 3$. Find the values of

(a) $\cot\theta$

Solution: From the identity $\cot\theta = \frac{1}{\tan\theta}$

$\Rightarrow$ $\cot\theta = \frac{1}{3}$

(b) $\sec\theta$

Solution: From the identity $1 + \tan^2\theta = \sec^2\theta$

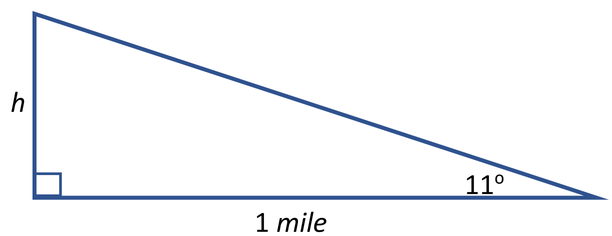
$\Rightarrow \quad \sec^2\theta = 1 + 3^2 = 1 + 9 = 10$

$\Rightarrow \quad \sec\theta = \sqrt{10}$

Examples:

[1] The angle of elevation to the top of the Empire State Building in New York is found to be 11° from the ground at a distance of 1 *mile* from the base of the building. Using this information, find the height of the Empire State Building.

Solution: Let *h* be the height of the building.

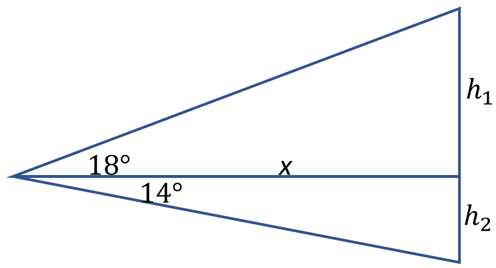


We can see that $\frac{h}{1} = \tan 11°$

$\Rightarrow \quad h = \tan 11° \approx 0.194380\, mile$

[2] A woman standing on a hill sees a flagpole that she knows is 60 *ft* tall. The angle of depression to the bottom of the pole is 14°, and the angle of elevation to the top of the pole is 18°. Find her distance *x* from the pole.

Solution:



We are told that $h_1 + h_2 = 60\ ft$ ... equation (1)

And we can see that $\frac{h_1}{x} = \tan 18° \quad \Rightarrow \quad h_1 = x \tan 18°$ ... equation (2)

$\frac{h_2}{x} = \tan 14° \quad \Rightarrow \quad h_2 = x \tan 14°$ ... equation (3)

Substituting (2) and (3) into (1),

$$x \tan 18° + x \tan 14° = 60$$

$$\Rightarrow \quad x(\tan 18° + \tan 14°) = 60$$

$$\Rightarrow \quad x = \frac{60}{\tan 18° + \tan 14°}\ ft$$

$$\Rightarrow \quad x \approx \frac{60}{0.324920 + 0.249328} \approx 104.484528\ ft$$

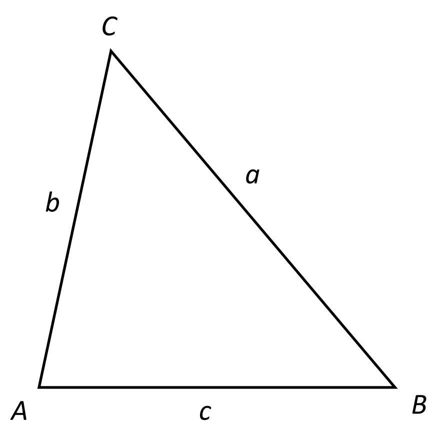
6.3 Trigonometric Functions of Angles, The Law of Sines and The Law of Cosines

Definitions:

- Angle θ is called an **acute** angle if $0° < \theta < 90°$
- Angle θ is called an **obtuse** angle if $90° < \theta < 180°$
- Angle θ is called a **reflex** angle if $180° < \theta < 360°$

Law of Sines

If ΔABC is a triangle with sides *a*, *b* and *c*, then



We have the Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

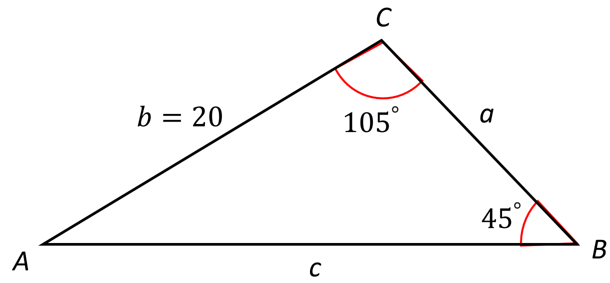
OR

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

And the area of ΔABC can be calculated from

$$Area = \frac{1}{2} bc \sin A = \frac{1}{2} ab \sin C = \frac{1}{2} ac \sin B$$

Example: Solve the unknowns and find the area of the triangle:



Solution: $\angle A + \angle B + \angle C = 180^\circ$

$\Rightarrow$ $\angle A + 45^\circ + 105^\circ = 180^\circ$

$\Rightarrow$ $\angle A = 180^\circ - 45^\circ - 105^\circ = 30^\circ$

Thus, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$\Rightarrow$ $\frac{a}{\sin 30^\circ} = \frac{20}{\sin 45^\circ} = \frac{c}{\sin 105^\circ}$

To find $\sin 105^\circ = \sin\left(60^\circ + 45^\circ\right)$

$= \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ$

$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{6}+\sqrt{2}}{4}$

So we have $\frac{a}{\frac{1}{2}} = \frac{20}{\frac{\sqrt{2}}{2}} = \frac{c}{\frac{\sqrt{6}+\sqrt{2}}{4}}$

$\frac{a}{\frac{1}{2}} = \frac{20}{\frac{\sqrt{2}}{2}}$

$\Rightarrow$ $2a = \frac{40}{\sqrt{2}}$

$\Rightarrow$ $a = \frac{20}{\sqrt{2}} = 20\left(\frac{\sqrt{2}}{2}\right) = 10\sqrt{2}$

And

$\frac{20}{\frac{\sqrt{2}}{2}} = \frac{c}{\frac{\sqrt{6}+\sqrt{2}}{4}}$

$\Rightarrow$ $\frac{4c}{\sqrt{6}+\sqrt{2}} = \frac{40}{\sqrt{2}}$

$\Rightarrow$ $c = 10\frac{\sqrt{6}+\sqrt{2}}{\sqrt{2}} = 10\left(\sqrt{3} + 1\right)$

The area of the triangle is

$$Area = \frac{1}{2} bc \sin A$$

$$= \frac{1}{2}(20)\left(10\left(\sqrt{3}+1\right)\right)\sin 30^{\circ}$$

$$= \frac{1}{2}(20)\left(10\left(\sqrt{3}+1\right)\right)\left(\frac{1}{2}\right) = 50\left(\sqrt{3}+1\right)\, unit^2$$

OR

$$Area = \frac{1}{2} ac \sin B$$

$$= \frac{1}{2}\left(10\sqrt{2}\right)\left(10\left(\sqrt{3}+1\right)\right)\sin 45^{\circ}$$

$$= \frac{1}{2}\left(10\sqrt{2}\right)\left(10\left(\sqrt{3}+1\right)\right)\left(\frac{\sqrt{2}}{2}\right) = 50\left(\sqrt{3}+1\right)\, unit^2$$

OR

$$Area = \frac{1}{2} ab \sin C$$

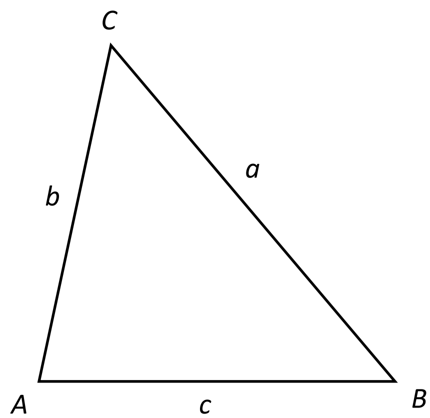
$$= \frac{1}{2}\left(10\sqrt{2}\right)(20)\sin 105^{\circ}$$

$$= \frac{1}{2}\left(10\sqrt{2}\right)(20)\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)$$

$$= \frac{1}{2}\left(10\sqrt{2}\right)(20)\left(\frac{\sqrt{2}\left(\sqrt{3}+1\right)}{4}\right) = 50\left(\sqrt{3}+1\right)\, unit^2$$

Law of Cosines

If ΔABC is a triangle with sides *a*, *b* and *c*, then



We have the Law of Cosines:

- $a^2 = b^2 + c^2 - 2bc\cos A$
- $b^2 = a^2 + c^2 - 2ac\cos B$
- $c^2 = a^2 + b^2 - 2ab\cos C$

Examples: Find two triangles for which $a = 2\ m$, $b = \left(\sqrt{6} - \sqrt{2}\right)m$ and $B = 15^{\circ}$

Solutions: $\frac{\sin A}{a} = \frac{\sin B}{b}$

$\Rightarrow$ $\frac{\sin A}{2} = \frac{\sin 15^{\circ}}{\sqrt{6}-\sqrt{2}}$

To find $\sin 15^{\circ}$, we could use

$$\sin 15^{\circ} = \sin\left(45^{\circ} - 30^{\circ}\right)$$

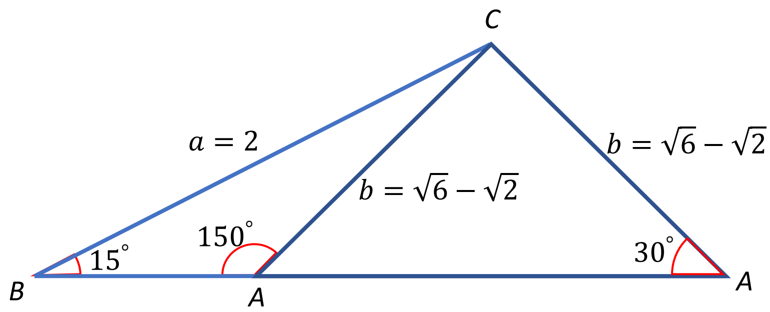
$$= \sin 45^{\circ}\cos 30^{\circ} - \cos 45^{\circ}\sin 30^{\circ}$$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$$

So we have $\frac{\sin A}{2} = \frac{\frac{\sqrt{6}-\sqrt{2}}{4}}{\sqrt{6}-\sqrt{2}}$

$\Rightarrow$ $\sin A = \frac{1}{2}$

$$\Rightarrow \qquad A = 30^{\circ} \text{ or } A = 150^{\circ}$$



There are two possible sizes for the angle $\angle A$. And angle $\angle C$ could either be

$$\angle C = 180^{\circ} - 15^{\circ} - 150^{\circ} = 15^{\circ}$$

OR

$$\angle C = 180^{\circ} - 15^{\circ} - 30^{\circ} = 135^{\circ}$$

Thus, there are also two possible lengths for the side *c*. From the Law of Cosines, we have

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\Rightarrow \qquad c^2 = 2^2 + \left(\sqrt{6} - \sqrt{2}\right)^2 - 2(2)\left(\sqrt{6} - \sqrt{2}\right)\cos 15^{\circ}$$

To find $\cos 15^{\circ}$, we could use

$$\cos 15^{\circ} = \cos\left(45^{\circ} - 30^{\circ}\right)$$

$$= \cos 45^{\circ}\cos 30^{\circ} + \sin 45^{\circ}\sin 30^{\circ}$$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{6}+\sqrt{2}}{4}$$

Thus, $\quad c^2 = 2^2 + \left(\sqrt{6} - \sqrt{2}\right)^2 - 2(2)\left(\sqrt{6} - \sqrt{2}\right)\frac{\sqrt{6}+\sqrt{2}}{4}$

$$= 4 + \left(\sqrt{6} - \sqrt{2}\right)^2 - \left(\sqrt{6} - \sqrt{2}\right)\left(\sqrt{6} + \sqrt{2}\right)$$

$$= 4 + \left(\sqrt{6} - \sqrt{2}\right)^2 - (6 - 2)$$

$$= 4 + \left(\sqrt{6} - \sqrt{2}\right)^2 - 4 = \left(\sqrt{6} - \sqrt{2}\right)^2$$

$$\therefore \qquad c = \left(\sqrt{6} - \sqrt{2}\right)\, units$$

OR

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$\Rightarrow \qquad c^2 = 2^2 + \left(\sqrt{6}-\sqrt{2}\right)^2 - 2(2)\left(\sqrt{6}-\sqrt{2}\right)\cos 135^{\circ}$

$$= 2^2 + \left(\sqrt{2}\left(\sqrt{3}-1\right)\right)^2 - 2(2)\sqrt{2}\left(\sqrt{3}-1\right)\left(-\frac{\sqrt{2}}{2}\right)$$

$$= 4 + 2\left(\sqrt{3}-1\right)^2 + 4\left(\sqrt{3}-1\right)$$

$$= 4 + 2\left(\sqrt{3}-1\right)\left(\sqrt{3}-1+2\right)$$

$$= 4 + 2\left(\sqrt{3}-1\right)\left(\sqrt{3}+1\right)$$

$$= 4 + 2(3-1) = 4 + 2(2) = 4 + 4 = 8$$

$\therefore \qquad c = 2\sqrt{2}\, units$

Or We could use the Law of Sines to find c

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

If $\angle C = 15^{\circ}$, then $\angle A = 150^{\circ}$ and we have

$$\frac{c}{\sin 15^{\circ}} = \frac{2}{\sin 150^{\circ}}$$

$\Rightarrow \qquad \frac{c}{\frac{\sqrt{6}-\sqrt{2}}{4}} = \frac{2}{\frac{1}{2}} = 4$

$\Rightarrow \qquad c = 4 \times \frac{\sqrt{6}-\sqrt{2}}{4} = \left(\sqrt{6}-\sqrt{2}\right) units$

OR If $\angle C = 135^{\circ}$, then $\angle A = 30^{\circ}$ and we have

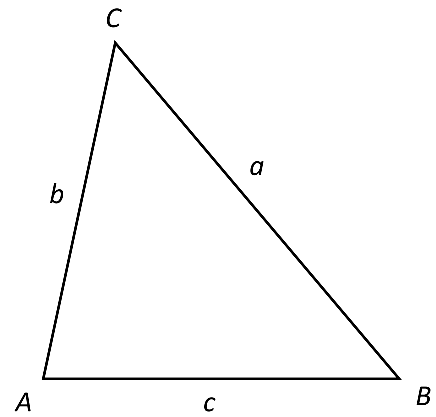
$$\frac{c}{\sin 135^{\circ}} = \frac{2}{\sin 30^{\circ}}$$

$\Rightarrow \qquad \frac{c}{\frac{\sqrt{2}}{2}} = \frac{2}{\frac{1}{2}} = 4$

$\Rightarrow \qquad c = 4 \times \frac{\sqrt{2}}{2} = 2\sqrt{2}\, units$

Heron's Area Formula

Given a triangle with sides *a*, *b* and *c*,



the area of the triangle can be calculated from Heron's formula

$$Area = \sqrt{s(s-a)(s-b)(s-c)}$$

Where $s = \frac{1}{2}(a+b+c)$

Example: Find the area of the triangle with sides $a = 11$, $b = 15$ and $c = 20$

Solution: Using Heron's formula,

$$Area = \sqrt{s(s-a)(s-b)(s-c)}$$

Where $s = \frac{1}{2}(a+b+c)$

$$= \frac{1}{2}(11+15+20) = \frac{1}{2} \times 46 = 23$$

Thus, $Area = \sqrt{s(s-a)(s-b)(s-c)}$

$$= \sqrt{23(23-11)(23-15)(23-20)}$$

$$= \sqrt{23 \times 12 \times 8 \times 3}$$

$$= \sqrt{23 \times (4 \times 3) \times (4 \times 2) \times 3}$$

$$= \sqrt{4^2 \times 3^2 \times 23 \times 2}$$

$$= (4 \times 3)\sqrt{23 \times 2} = 12\sqrt{46}\, unit^2$$

7.1 Trigonometric Identities

Reciprocal Identities:

- $\csc x = \frac{1}{\sin x}$
- $\sec x = \frac{1}{\cos x}$
- $\cot x = \frac{1}{\tan x}$

Quotient Identities:

- $\tan x = \frac{\sin x}{\cos x}$
- $\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$

Pythagorean Identities:

- $\sin^2 x + \cos^2 x = 1$
- $1 + \tan^2 x = \sec^2 x$
- $1 + \cot^2 x = \csc^2 x$

Co-function Identities:

- $\sin\left(\frac{\pi}{2} - x\right) = \cos x$
- $\cos\left(\frac{\pi}{2} - x\right) = \sin x$
- $\tan\left(\frac{\pi}{2} - x\right) = \cot x$
- $\cot\left(\frac{\pi}{2} - x\right) = \tan x$

Even/Odd Identities:

- $\sin(-x) = -\sin x$, i.e. $\sin x$ is an odd function.
- $\cos(-x) = \cos x$, i.e. $\cos x$ is an even function.
- $\tan(-x) = -\tan x$, i.e. $\tan x$ is an odd function.

In the following topics, we shall use the above identities to verify more complicated identities and to solve trigonometric equations.

Examples: Simplify the following expressions:

1. $\tan^2 x - \tan^2 x \sin^2 x = \tan^2 x\,(1 - \sin^2 x)$

$$= \frac{\sin^2 x}{\cos^2 x} \cdot \cos^2 x = \sin^2 x$$

2. $\tan^4 x + 2\tan^2 x + 1 = (\tan^2 x + 1)^2$

$$= (\sec^2 x)^2 = \sec^4 x$$

3. $(\sin x + \cos x)^2 = \sin^2 x + 2\sin x \cos x + \cos^2 x$

$$= \sin^2 x + \cos^2 x + 2\sin x \cos x$$

$= 1 + \sin(2x)$, We will be discussing double angle formulae in a later topic.

Note: From the identity $\sin^2 x + \cos^2 x = 1$, we can obtain the following two identities

- $1 - \cos^2 x = \sin^2 x$
- $1 - \sin^2 x = \cos^2 x$

4. $\frac{1}{1+\cos x} + \frac{1}{1-\cos x} = \frac{1-\cos x}{(1+\cos x)(1-\cos x)} + \frac{1+\cos x}{(1+\cos x)(1-\cos x)}$

$$= \frac{(1-\cos x)+(1+\cos x)}{(1+\cos x)(1-\cos x)}$$

$$= \frac{2}{1-\cos^2 x}$$

$$= \frac{2}{\sin^2 x} = 2\csc^2 x$$

5. $\frac{\sin^2 y}{1-\cos y} = \frac{\sin^2 y}{1-\cos y} \cdot \frac{1+\cos y}{1+\cos y}$, Multiplying and dividing by the conjugate of the denominator.

$$= \frac{\sin^2 y(1+\cos y)}{1-\cos^2 y}$$

$$= \frac{\sin^2 y(1+\cos y)}{\sin^2 y} = 1 + \cos y$$

OR

$$\frac{\sin^2 y}{1-\cos y} = \frac{1-\cos^2 y}{1-\cos y}$$

$$= \frac{(1-\cos y)(1+\cos y)}{1-\cos y} = 1 + \cos y$$

6. $\ln|\cos x| - \ln|\sin x| = \ln\left|\frac{\cos x}{\sin x}\right| = \ln|\cot x|$

Use the trigonometric substitution to write the algebraic expression as a trigonometric function of θ, where $0 < \theta < \frac{\pi}{2}$.

7. $\sqrt{9 - x^2}$, $x = 3\cos\theta$

$$= \sqrt{9 - (3\cos\theta)^2}$$

$$= \sqrt{9 - 9\cos^2\theta}$$

$$= \sqrt{9(1 - \cos^2\theta)}$$

$$= \sqrt{9\sin^2\theta} = 3\sin\theta$$

Examples: Verify the identity:

8. $\cos^2\beta - \sin^2\beta = 1 - 2\sin^2\beta$

We shall start with $\cos^2\beta - \sin^2\beta$ as it is the more complicated side.

$$\cos^2\beta - \sin^2\beta = 1 - \sin^2\beta - \sin^2\beta = 1 - 2\sin^2\beta$$

9. $\frac{1}{\cos x+1}+\frac{1}{\cos x-1}=-2\csc x\cot x$

As before, we always begin from the more complicated side

$$\frac{1}{\cos x+1}+\frac{1}{\cos x-1}=\frac{(\cos x-1)+(\cos x+1)}{(\cos x+1)(\cos x-1)}$$

$$=\frac{2\cos x}{\cos^2 x-1}$$

$$=\frac{2\cos x}{-(1-\cos^2 x)}$$

$$=\frac{2\cos x}{-\sin^2 x}$$

$$=-2\cdot\frac{1}{\sin x}\cdot\frac{\cos x}{\sin x}=-2\csc x\cot x$$

10. $(1+\cot^2 x)\cos^2 x=\cot^2 x$

$\Rightarrow$ $(1+\cot^2 x)\cos^2 x=\left(1+\frac{\cos^2 x}{\sin^2 x}\right)\cos^2 x$

$$=\frac{\sin^2 x+\cos^2 x}{\sin^2 x}\cdot\cos^2 x$$

$$=\frac{1}{\sin^2 x}\cdot\cos^2 x=\frac{\cos^2 x}{\sin^2 x}=\cot^2 x$$

11. $\csc x-\sin x=\cos x\cot x$

$\Rightarrow$ $\cos x\cot x=\cos x\frac{\cos x}{\sin x}$

$$=\frac{\cos^2 x}{\sin x}$$

$$=\frac{1-\sin^2 x}{\sin x}$$

$$=\frac{1}{\sin x}-\frac{\sin^2 x}{\sin x}$$

$$=\frac{1}{\sin x}-\sin x=\csc x-\sin x$$

12. $\frac{1+\cos x}{\sin x} = \frac{\sin x}{1-\cos x}$

$\Rightarrow$ $\frac{1+\cos x}{\sin x} = \frac{1+\cos x}{\sin x} \cdot \frac{1-\cos x}{1-\cos x}$

$$= \frac{1-\cos^2 x}{\sin x(1-\cos x)}$$

$$= \frac{\sin^2 x}{\sin x(1-\cos x)} = \frac{\sin x}{1-\cos x}$$

13. $\frac{\cot^2 t}{\csc t} = \frac{1-\sin^2 t}{\sin t}$

$\Rightarrow$ $\frac{\cot^2 t}{\csc t} = \frac{\frac{\cos^2 t}{\sin^2 t}}{\frac{1}{\sin t}}$

$$= \frac{\cos^2 t}{\sin^2 t} \times \frac{\sin t}{1}$$

$$= \frac{\cos^2 t}{\sin t} = \frac{1-\sin^2 t}{\sin t}$$

14. $\cos^3 x \sin^2 x = (\sin^2 x - \sin^4 x) \cos x$

$\Rightarrow$ $(\sin^2 x - \sin^4 x) \cos x = \sin^2 x\,(1 - \sin^2 x) \cos x$

$$= \sin^2 x \cos^2 x \cos x$$

$$= \cos^3 x \sin^2 x$$

7.2 Addition and Subtraction Formulas

Memorise the following formulas:

- $\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$
- $\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$
- $\tan(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u \tan v}$

Examples:

1. $\sin 75^\circ = \sin(30^\circ + 45^\circ)$

$$= \sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ$$

$$= \frac{1}{2} \times \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{\sqrt{2}+\sqrt{6}}{4}$$

2. $\cos 105^\circ = \cos(45^\circ + 60^\circ)$

$$= \cos 45^\circ \cos 60^\circ - \sin 45^\circ \sin 60^\circ$$

$$= \frac{\sqrt{2}}{2} \times \frac{1}{2} - \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \frac{\sqrt{2}-\sqrt{6}}{4}$$

More identities:

- $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$

Proof: We let $\theta = \sin^{-1} x$,

$\Rightarrow \quad \sin \theta = x$

$\Rightarrow \quad \sin^2 \theta = x^2$

$\Rightarrow \quad 1 - \cos^2 \theta = x^2$

$\Rightarrow \quad \cos^2 \theta = 1 - x^2$

$\Rightarrow \quad \cos \theta = \pm\sqrt{1 - x^2}$

$\Rightarrow \qquad \cos(\sin^{-1} x) = \pm\sqrt{1 - x^2}$, since $\theta = \sin^{-1} x$

Since $-\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2}$, so we have $0 \leq \cos(\sin^{-1} x) \leq 1$, and hence

$$\cos(\sin^{-1} x) = +\sqrt{1 - x^2}$$

- $\sin(\cos^{-1} x) = \sqrt{1 - x^2}$

Proof: We let $\varphi = \cos^{-1} x$

$\Rightarrow \qquad \cos\varphi = x$

$\Rightarrow \qquad \cos^2\varphi = x^2$

$\Rightarrow \qquad 1 - \sin^2\varphi = x^2$

$\Rightarrow \qquad \sin^2\varphi = 1 - x^2$

$\Rightarrow \qquad \sin\varphi = \pm\sqrt{1 - x^2}$

$\Rightarrow \qquad \sin(\cos^{-1} x) = \pm\sqrt{1 - x^2}$, since $\varphi = \cos^{-1} x$

Since $0 \leq \cos^{-1} x \leq \pi$, so we have $0 \leq \sin(\cos^{-1} x) \leq 1$, and hence

$$\sin(\cos^{-1} x) = +\sqrt{1 - x^2}$$

Evaluating Expressions Involving Inverse Trigonometric Functions

Example:

3. $\sin(\sin^{-1} x + \cos^{-1} x) = \sin(\sin^{-1} x)\cos(\cos^{-1} x) + \cos(\sin^{-1} x)\sin(\cos^{-1} x)$

$$= (x)(x) + \left(\sqrt{1 - x^2}\right)\left(\sqrt{1 - x^2}\right)$$

$$= x^2 + 1 - x^2 = 1$$

More examples:

4. $\sin\left(x+\frac{\pi}{6}\right)-\sin\left(x-\frac{\pi}{6}\right)=\frac{1}{2}$

$\Rightarrow \quad \left(\sin x\cos\frac{\pi}{6}+\cos x\sin\frac{\pi}{6}\right)-\left(\sin x\cos\frac{\pi}{6}-\cos x\sin\frac{\pi}{6}\right)=\frac{1}{2}$

$\Rightarrow \quad \sin x\cos\frac{\pi}{6}+\cos x\sin\frac{\pi}{6}-\sin x\cos\frac{\pi}{6}+\cos x\sin\frac{\pi}{6}=\frac{1}{2}$

$\Rightarrow \quad 2\cos x\sin\frac{\pi}{6}=\frac{1}{2}$

$\Rightarrow \quad 2(\cos x)\left(\frac{1}{2}\right)=\frac{1}{2}$

$\Rightarrow \quad \cos x=\frac{1}{2}$

$\therefore \quad x=\frac{\pi}{3}+2n\pi$ or $x=\frac{5\pi}{3}+2n\pi$

$\Rightarrow \quad x=\left(\frac{1}{3}+2n\right)\pi$ or $x=\left(\frac{5}{3}+2n\right)\pi$, where *n* is an integer.

The following example is used in calculus to obtain the derivative of $\cos x$:

5.
$$\begin{aligned}
\frac{\cos(x+h)-\cos x}{h} &= \frac{\cos x\cos h-\sin x\sin h-\cos x}{h}\\
&= \frac{\cos x\cos h-\cos x-\sin x\sin h}{h}\\
&= \frac{\cos x(\cos h-1)-\sin x\sin h}{h}\\
&= \frac{\cos x(\cos h-1)}{h}-\frac{\sin x\sin h}{h}\\
&= \cos x\frac{\cos h-1}{h}-\sin x\frac{\sin h}{h}
\end{aligned}$$

Sums of Sines and Cosines

If *A* and *B* are real numbers, then

$$A\sin x+B\cos x=k\sin(x+\emptyset)$$

Where $k=\sqrt{A^2+B^2}$

And $\quad \emptyset=\cos^{-1}\frac{A}{\sqrt{A^2+B^2}}=\sin^{-1}\frac{B}{\sqrt{A^2+B^2}}$

Examples: Rewrite the following expressions in terms of only sine.

[1] $\sin x + \cos x = 1\sin x + 1\cos x$

$$= \sqrt{1^2+1^2}\left(\frac{1}{\sqrt{1^2+1^2}}\sin x + \frac{1}{\sqrt{1^2+1^2}}\cos x\right)$$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)$$

$= \sqrt{2}(\sin x\cos 45° + \cos x \sin 45°)$; We know that $\sin 45° = \frac{1}{\sqrt{2}}$ and $\cos 45° = \frac{1}{\sqrt{2}}$.

$$= \sqrt{2}\sin(x + 45°)$$

[2] $3\sin(\pi x) + 3\sqrt{3}\cos(\pi x) = \sqrt{3^2+\left(3\sqrt{3}\right)^2}\left(\frac{3}{\sqrt{3^2+\left(3\sqrt{3}\right)^2}}\sin(\pi x) + \frac{3\sqrt{3}}{\sqrt{3^2+\left(3\sqrt{3}\right)^2}}\cos(\pi x)\right)$

$$= \sqrt{9+(9)(3)}\left(\frac{3}{\sqrt{9+(9)(3)}}\sin(\pi x) + \frac{3\sqrt{3}}{\sqrt{9+(9)(3)}}\cos(\pi x)\right)$$

$$= \sqrt{9+27}\left(\frac{3}{\sqrt{9+27}}\sin(\pi x) + \frac{3\sqrt{3}}{\sqrt{9+27}}\cos(\pi x)\right)$$

$$= \sqrt{36}\left(\frac{3}{\sqrt{36}}\sin(\pi x) + \frac{3\sqrt{3}}{\sqrt{36}}\cos(\pi x)\right)$$

$$= 6\left(\frac{3}{6}\sin(\pi x) + \frac{3\sqrt{3}}{6}\cos(\pi x)\right)$$

$$= 6\left(\frac{1}{2}\sin(\pi x) + \frac{\sqrt{3}}{2}\cos(\pi x)\right)$$

$= 6(\sin(\pi x)\cos 60° + \cos(\pi x)\sin 60°)$; $\cos 60° = \frac{1}{2}$ and $\sin 60° = \frac{\sqrt{3}}{2}$

$$= 6\sin(\pi x + 60°)$$

7.3 Double-Angle, Half-Angle and Product-Sum Formulas

Multiple-Angle and Product-to-Sum Formulas

Recall from the previous topic the following formulas:

- $\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$
- $\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$
- $\tan(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u \tan v}$

If $u = v$, we obtain the **double-angle formulas**:

- $\sin(2u) = 2 \sin u \cos u$
- $\cos(2u) = \cos^2 u - \sin^2 u$

  $= 2\cos^2 u - 1$

  $= 1 - 2\sin^2 u$
- $\tan(2u) = \frac{2 \tan u}{1 - \tan^2 u}$

Examples:

1. $\sin(2x) - \sin x = 0$

$\Rightarrow$ $2 \sin x \cos x - \sin x = 0$

$\Rightarrow$ $\sin x\,(2\cos x - 1) = 0$

$\Rightarrow$ $\sin x = 0$ or $2\cos x - 1 = 0$

$\Rightarrow$ $\sin x = 0$ or $2\cos x = 1$

$\Rightarrow$ $\sin x = 0$ or $\cos x = \frac{1}{2}$

$\therefore$ $x = n\pi$ or $x = \frac{\pi}{3} + 2n\pi$ or $x = \frac{5\pi}{3} + 2n\pi$

$\Rightarrow$ $x = n\pi$ or $x = \left(\frac{1}{3} + 2n\right)\pi$ or $x = \left(\frac{5}{3} + 2n\right)\pi$, where *n* is an integer.

2. $\cos(3x) = \cos(2x + x)$

$$= \cos(2x)\cos x - \sin(2x)\sin x$$

$$= (2\cos^2 x - 1)\cos x - (2\sin x\cos x)\sin x$$

$$= 2\cos^3 x - \cos x - 2\sin^2 x\cos x$$

$$= 2\cos^3 x - \cos x - 2(1 - \cos^2 x)\cos x$$

$$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$$

$$= 4\cos^3 x - 3\cos x$$

$\therefore$ $\cos(3x) = 4\cos^3 x - 3\cos x$

Half-Angle Formulas:

From the Cosine double angle formulas, we can obtain the **half-angle formulas**:

- $\cos(2u) = 1 - 2\sin^2 u$

$\Rightarrow$ $2\sin^2 u = 1 - \cos(2u)$

$\Rightarrow$ $\sin^2 u = \frac{1-\cos(2u)}{2}$

$\Rightarrow$ $\sin u = \pm\sqrt{\frac{1-\cos(2u)}{2}}$

- $\cos(2u) = 2\cos^2 u - 1$

$\Rightarrow$ $2\cos^2 u = 1 + \cos(2u)$

$\Rightarrow$ $\cos^2 u = \frac{1+\cos(2u)}{2}$

$\Rightarrow$ $\cos u = \pm\sqrt{\frac{1+\cos(2u)}{2}}$

- $\tan u = \frac{\sin u}{\cos u}$

$$= \frac{\sqrt{\frac{1-\cos(2u)}{2}}}{\sqrt{\frac{1+\cos(2u)}{2}}}$$

$$= \sqrt{\frac{1-\cos(2u)}{1+\cos(2u)}}$$

Two things can happen here:

$$\tan u = \sqrt{\frac{1-\cos(2u)}{1+\cos(2u)} \times \frac{1-\cos(2u)}{1-\cos(2u)}}$$

$$= \sqrt{\frac{(1-\cos(2u))^2}{1-\cos^2(2u)}}$$

$$= \sqrt{\frac{(1-\cos(2u))^2}{\sin^2(2u)}}$$

$$= \frac{1-\cos(2u)}{\sin(2u)}$$

OR

$$\tan u = \sqrt{\frac{1-\cos(2u)}{1+\cos(2u)} \times \frac{1+\cos(2u)}{1+\cos(2u)}}$$

$$= \sqrt{\frac{1-\cos^2(2u)}{(1+\cos(2u))^2}}$$

$$= \sqrt{\frac{\sin^2(2u)}{(1+\cos(2u))^2}}$$

$$= \frac{\sin(2u)}{1+\cos(2u)}$$

In all the above, when we replace u with $\frac{u}{2}$ (or $2u$ with u), we obtain the half-angle formulas:

- $\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}}$
- $\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$
- $\tan\frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{\sin u}{1+\cos u}$

Examples:

3. $\cos 105^\circ = -\sqrt{\frac{1+\cos(2\times 105^\circ)}{2}}$

$= -\sqrt{\frac{1+\cos 210^\circ}{2}}$, We take the negative sign because Cosine is negative in Quadrant II.

$= -\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = -\sqrt{\frac{2-\sqrt{3}}{4}} = -\frac{\sqrt{2-\sqrt{3}}}{2}$

4. $\tan 165^\circ = \frac{1-\cos(2\times 165^\circ)}{\sin(2\times 165^\circ)}$

$= \frac{1-\cos 330^\circ}{\sin 330^\circ}$

$= \frac{1-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\frac{\frac{2-\sqrt{3}}{2}}{\frac{1}{2}} = -(2-\sqrt{3})$

OR

$\tan 165^\circ = \frac{\sin(2\times 165^\circ)}{1+\cos(2\times 165^\circ)}$

$= \frac{\sin 330^\circ}{1+\cos 330^\circ}$

$= \frac{-\frac{1}{2}}{1+\frac{\sqrt{3}}{2}} = \frac{-\frac{1}{2}}{\frac{2+\sqrt{3}}{2}} = -\frac{1}{2+\sqrt{3}} = -\frac{2-\sqrt{3}}{(2+\sqrt{3})(2-\sqrt{3})} = -\frac{2-\sqrt{3}}{4-3} = -\frac{2-\sqrt{3}}{1} = -(2-\sqrt{3})$

The following formulas are very important in calculus, especially when we want to find the antiderivatives of trigonometric products.

Product-to-Sum Formulas

- $\sin u \sin v = -\frac{1}{2}\ (\cos(u+v) - \cos(u-v))$
- $\cos u \cos v = \frac{1}{2}\ (\cos(u+v) + \cos(u-v))$
- $\sin u \cos v = \frac{1}{2}\ (\sin(u+v) + \sin(u-v))$
- $\cos u \sin v = \frac{1}{2}\ (\sin(u+v) - \sin(u-v))$

Examples:

5. $\sin\frac{\pi}{3}\cos\frac{\pi}{6} = \frac{1}{2}\left(\sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{3}-\frac{\pi}{6}\right) \right)$

$$= \frac{1}{2}\left(\sin\left(\frac{2\pi}{6}+\frac{\pi}{6}\right) + \sin\left(\frac{2\pi}{6}-\frac{\pi}{6}\right) \right)$$

$$= \frac{1}{2}\left(\sin\frac{3\pi}{6} + \sin\frac{\pi}{6} \right) = \frac{1}{2}\left(\sin\frac{\pi}{2} + \sin\frac{\pi}{6} \right)$$

$$= \frac{1}{2}\left(1 + \frac{1}{2} \right) = \frac{1}{2} \times \frac{3}{2} = \frac{3}{4}$$

6. Rewrite as a sum of trigonometric functions:

$$\sin(5\theta)\sin(3\theta) = -\frac{1}{2}(\cos(5\theta+3\theta) - \cos(5\theta-3\theta))$$

$$= -\frac{1}{2}(\cos(8\theta) - \cos(2\theta))$$

The following **Sum-to-Product Formulas** can be obtained from the previous Product-to-Sum Formulas:

- $\sin u + \sin v = 2\sin\frac{u+v}{2}\cos\frac{u-v}{2}$
- $\sin u - \sin v = 2\cos\frac{u+v}{2}\sin\frac{u-v}{2}$
- $\cos u + \cos v = 2\cos\frac{u+v}{2}\cos\frac{u-v}{2}$
- $\cos u - \cos v = -2\sin\frac{u+v}{2}\sin\frac{u-v}{2}$

Examples:

7. $\sin 75^\circ + \sin 15^\circ = 2\sin\frac{75^\circ+15^\circ}{2}\cos\frac{75^\circ-15^\circ}{2}$

$$= 2\sin\frac{90^\circ}{2}\cos\frac{60^\circ}{2}$$

$$= 2\sin 45^\circ\cos 30^\circ$$

$$= 2 \times \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{6}}{2}$$

OR

$$\sin 75^\circ + \sin 15^\circ = \sin\left(45^\circ + 30^\circ\right) + \sin\left(45^\circ - 30^\circ\right)$$

$$= \left(\sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ\right) + \left(\sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ\right)$$

$$= 2 \sin 45^\circ \cos 30^\circ$$

$$= 2 \times \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{6}}{2}$$

8. Simplify $\frac{\cos(4x)+\cos(2x)}{\sin(4x)+\sin(2x)} = \frac{2\cos\frac{4x+2x}{2}\cos\frac{4x-2x}{2}}{2\sin\frac{4x+2x}{2}\cos\frac{4x-2x}{2}}$

$$= \frac{\cos\frac{6x}{2}}{\sin\frac{6x}{2}} = \frac{\cos(3x)}{\sin(3x)} = \cot(3x)$$

7.4 Basic and More Trigonometric Equations

Remember to:

1. Treat each trigonometric function like you would treat a variable.

2. $\sin x = \sin(x + 2n\pi)$, the period of $\sin x$ is 2π

   $\cos x = \cos(x + 2n\pi)$, the period of $\cos x$ is 2π

   And $\tan x = \tan(x + n\pi)$, the period of $\tan x$ is $n\pi$

   Where *n* is an integer.

Examples:

1. $\sin x(\sin x + 1) = 0$

$\Rightarrow$ $\sin x = 0$ or $\sin x + 1 = 0$

$\Rightarrow$ $\sin x = 0$ or $\sin x = -1$

When $\sin x = 0$, we have

$x = 0, \pm\pi, \pm2\pi, \pm3\pi, \pm4\pi, \cdots$

$\Rightarrow$ $x = n\pi$, where *n* is an integer.

When $\sin x = -1$, we have

$\Rightarrow$ $x = \frac{3}{2}\pi + 2n\pi = \left(\frac{3}{2} + 2n\right)\pi$, where *n* is an integer.

$\therefore$ $x = n\pi$ or $x = \left(\frac{3}{2} + 2n\right)\pi$, where *n* is an integer.

2. $\sin^2 x = 3\cos^2 x$

We could choose to use the identity $\sin^2 x = 1 - \cos^2 x$,

$\sin^2 x = 3\cos^2 x$

$\Rightarrow$ $1 - \cos^2 x = 3\cos^2 x$

$\Rightarrow$ $4\cos^2 x = 1$

$\Rightarrow \quad \cos^2 x = \frac{1}{4}$

$\Rightarrow \quad \cos x = \frac{1}{2}$ or $\cos x = -\frac{1}{2}$

$\therefore \quad x = \frac{\pi}{3} + n\pi$ or $x = \frac{2\pi}{3} + n\pi$, where *n* is an integer.

OR We could choose to use the identity $\cos^2 x = 1 - \sin^2 x$

$\sin^2 x = 3\cos^2 x$

$\Rightarrow \quad \sin^2 x = 3(1 - \sin^2 x)$

$\Rightarrow \quad \sin^2 x = 3 - 3\sin^2 x$

$\Rightarrow \quad 4\sin^2 x = 3$

$\Rightarrow \quad \sin^2 x = \frac{3}{4}$

$\Rightarrow \quad \sin x = \frac{\sqrt{3}}{2}$ or $\sin x = -\frac{\sqrt{3}}{2}$

$\therefore \quad x = \frac{\pi}{3} + n\pi$ or $x = \frac{2\pi}{3} + n\pi$, where *n* is an integer.

$\Rightarrow \quad x = \left(\frac{1}{3} + n\right)\pi$ or $x = \left(\frac{2}{3} + n\right)\pi$, where *n* is an integer.

3. $2\cos^2 x + \cos x - 1 = 0$

$\Rightarrow \quad (2\cos x - 1)(\cos x + 1) = 0$

$\Rightarrow \quad 2\cos x - 1 = 0$ or $\cos x + 1 = 0$

$\Rightarrow \quad 2\cos x = 1$ or $\cos x = -1$

$\Rightarrow \quad \cos x = \frac{1}{2}$ or $\cos x = -1$

$\therefore \quad x = \frac{\pi}{3} + 2n\pi$ or $x = \frac{5\pi}{3} + 2n\pi$ or $x = \pi + 2n\pi$, where *n* is an integer.

$\Rightarrow \quad x = \left(\frac{1}{3} + 2n\right)\pi$ or $x = \left(\frac{5}{3} + 2n\right)\pi$ or $x = (1 + 2n)\pi$, where *n* is an integer.

4. $\sec x + \tan x = 1$

$\Rightarrow \quad \frac{1}{\cos x} + \frac{\sin x}{\cos x} = 1$

$\Rightarrow \quad 1 + \sin x = \cos x$

$\Rightarrow \quad (1 + \sin x)^2 = \cos^2 x$

$\Rightarrow \quad 1 + 2\sin x + \sin^2 x = 1 - \sin^2 x$

$\Rightarrow \quad 1 + 2\sin x + \sin^2 x - 1 + \sin^2 x = 0$

$\Rightarrow \quad 2\sin^2 x + 2\sin x = 0$

$\Rightarrow \quad 2\sin x\,(\sin x + 1) = 0$

$\Rightarrow \quad \sin x = 0$ or $\sin x + 1 = 0$

$\Rightarrow \quad \sin x = 0$ or $\sin x = -1$

$\therefore \quad x = n\pi$ or $x = \frac{3}{2}\pi + 2n\pi$, where *n* is an integer.

$\Rightarrow \quad x = n\pi$ or $x = \left(\frac{3}{2} + 2n\right)\pi$, where *n* is an integer.

5. $\cos(2x) = \frac{1}{2}$

$\Rightarrow \quad 2x = \frac{\pi}{3} + 2n\pi$ or $2x = \frac{5}{3}\pi + 2n\pi$, where *n* is an integer.

$\therefore \quad x = \frac{\pi}{6} + n\pi$ or $x = \frac{5}{6}\pi + n\pi$, where *n* is an integer.

$\Rightarrow \quad x = \left(\frac{1}{6} + n\right)\pi$ or $x = \left(\frac{5}{6} + n\right)\pi$, where *n* is an integer.

6. $\tan(3x) = 1$

$\Rightarrow \quad 3x = \frac{\pi}{4} + n\pi$, where *n* is an integer.

$\therefore \quad x = \frac{\pi}{12} + \frac{n}{3}\pi$, where *n* is an integer.

$\Rightarrow \quad x = \left(\frac{1}{12} + \frac{n}{3}\right)\pi$, where *n* is an integer.

7. $\cos\frac{x}{2} = \frac{\sqrt{2}}{2}$

$\Rightarrow \quad \frac{x}{2} = \frac{\pi}{4} + 2n\pi$ or $\frac{x}{2} = \frac{7}{4}\pi + 2n\pi$

∴ $x = \frac{\pi}{2} + 4n\pi$ or $x = \frac{7}{2}\pi + 4n\pi$, where *n* is an integer.

⇒ $x = \left(\frac{1}{2} + 4n\right)\pi$ or $x = \left(\frac{7}{2} + 4n\right)\pi$, where *n* is an integer.