Name: KEY

Business Calculus

Final Exam

Instructions: Complete all problems, showing all relevant work. Reduce or simplify answers as much as possible without using a calculator. Circle or clearly mark your final answers.

1. [5 points each] Calculate the limits.

a. $\lim_{x \to 6} \frac{x^2 - 36}{x^2 - 8x + 12}$ $= \lim_{x \to 6} \frac{(x+6)\cancel{(x-6)}}{(x-2)\cancel{(x-6)}} = \frac{12}{4} = \boxed{3}$

b. $\lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$ $= \frac{d}{dx}(x^3) = \boxed{3x^2}$

2. [10 points] Use the <u>limit</u> <u>definition</u> of derivative to find $\frac{d}{dx}[x^2]$. You must use the limit definition to receive any points.

$$\frac{d}{dx}(x^2) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} = \boxed{2x}$$

3. [5 points each] Find the derivatives of the functions.

a. $f(x) = \dfrac{x^2}{x+1}$

$$f'(x) = \frac{(x+1)(2x) - x^2}{(x+1)^2} = \frac{2x^2 + 2x - x^2}{(x+1)^2} = \boxed{\frac{x^2+2x}{(x+1)^2}}$$

b. $g(x) = x\ln(x) + e^{x^2+x}$

$$g'(x) = 1 \cdot \ln(x) + x\frac{1}{x} + e^{x^2+x}(2x+1)$$

$$= \boxed{\ln x + 1 + (2x+1)e^{x^2+x}}$$

4. [10 points] Find dy/dx for the equation: $xy = x^2 + \ln(y^2)$

$$x\frac{dy}{dx} + y = 2x + \frac{2}{y}\frac{dy}{dx}$$

$$\left(x - \frac{2}{y}\right)\frac{dy}{dx} = 2x - y \quad \Rightarrow \quad \frac{dy}{dx} = \boxed{\frac{2xy - y^2}{xy - 2}}$$

5. [10 points] **a.** Find an equation for the tangent line to $y = \sqrt{x}$ at the point $(25, 5)$.

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}, \text{ at } (25,5): \frac{dy}{dx} = m = \frac{1}{2\sqrt{25}} = \frac{1}{10}$$

line: $y - 5 = \frac{1}{10}(x - 25) \Rightarrow y = \frac{1}{10}x - \frac{5}{2} + 5 \Rightarrow \boxed{y = \frac{1}{10}x + \frac{5}{2}}$

b. Use part **a.** to estimate $\sqrt{25.1}$.

$$\sqrt{25.1} \approx \frac{1}{10}\cdot(25.1) + \frac{5}{2} = \frac{1}{10}\cdot\frac{251}{10} + \frac{5}{2} = \frac{251}{100} + \frac{250}{100} = \frac{501}{100}$$

$$= \boxed{5.01}$$

6. [10 points] A clown is blowing up a spherical balloon. When the volume of the balloon is $\frac{32}{3}\pi$ cm^3, the surface area is changing at a rate of 1 cm^2/sec. At what rate is the radius increasing at this moment? [volume: $V = \frac{4}{3}\pi r^3$, surface area: $A = 4\pi r^2$]

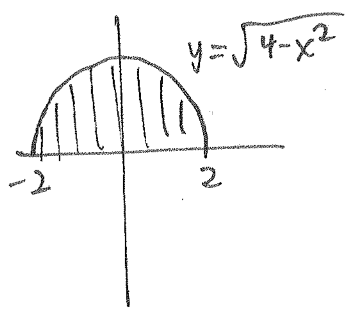
$$V = \frac{32}{\cancel{3}}\cancel{\pi} = \frac{4}{\cancel{3}}\cancel{\pi} r^3 \Rightarrow r^3 = 8 \Rightarrow r = 2$$

$$\frac{dA}{dt} = 8\pi r \frac{dr}{dt}$$

$$1 = 8\pi(2)\frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \boxed{\frac{1}{16\pi} \text{ cm/sec}}$$

7. [10 points] Calculate: $\int_{-2}^{2} \sqrt{4-x^2}\, dx$



$$\text{Area} = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi \cdot 4 = \boxed{2\pi}$$

8. [10 points] Find the average value of the function $f(x) = x^2 + x$ between $x = -1$ and $x = 2$.

$$f_{avg} = \frac{1}{2+1}\int_{-1}^{2} x^2 + x \, dx = \frac{1}{3}\left(\frac{1}{3}x^3 + \frac{1}{2}x^2\right)\Big|_{-1}^{2}$$

$$= \frac{1}{3}\left(\frac{1}{3}(8) + 2\right) - \frac{1}{3}\left(-\frac{1}{3} + \frac{1}{2}\right)$$

$$= \frac{8}{9} + \frac{2}{3} + \frac{1}{9} - \frac{1}{6} = 1 + \frac{1}{2} = \boxed{\frac{3}{2}}$$

9. [5 points each] Find the (definite and indefinite) integrals.

a. $\int x^4 + 5x - \frac{1}{x} + \frac{3}{x^2}\, dx = \boxed{\frac{1}{5}x^5 + \frac{5}{2}x^2 - \ln|x| - \frac{3}{x} + C}$

b. $\cancel{\int_1^e 4t^3 \ln(t^4)\, dt = \int_1^{e^4} \ln u \, du}$

$\cancel{u = t^4 \quad u(e) = e^4}$

$\cancel{du = 4t^3 dt \quad u(1) = 1}$

$\int_1^e \frac{4t^3}{t^4}\, dt = 4\int_1^e \frac{1}{t}\, dt$

$= 4 \ln(t) \Big|_1^e$

$= 4(\ln e - \ln 1)$

$= \boxed{4}$

c. $\frac{1}{2}\int_0^4 \frac{2x}{\sqrt{x^2+9}}\, dx = \frac{1}{2}\int_9^{25} u^{-1/2}\, du = \frac{1}{2}\left(2u^{1/2}\right)\Big|_9^{25} = 5 - 3 = \boxed{2}$

$u = x^2 + 9 \quad u(4) = 16 + 9 = 25$

$du = 2x\, dx \quad u(0) = 9$

d. $-\int \frac{1 - \ln(x)}{-x}\, dx = -\int u\, du = -\frac{1}{2}u^2 + C$

$u = 1 - \ln(x)$

$du = \frac{-1}{x}\, dx$

$= \boxed{-\frac{1}{2}(1 - \ln(x))^2 + C}$