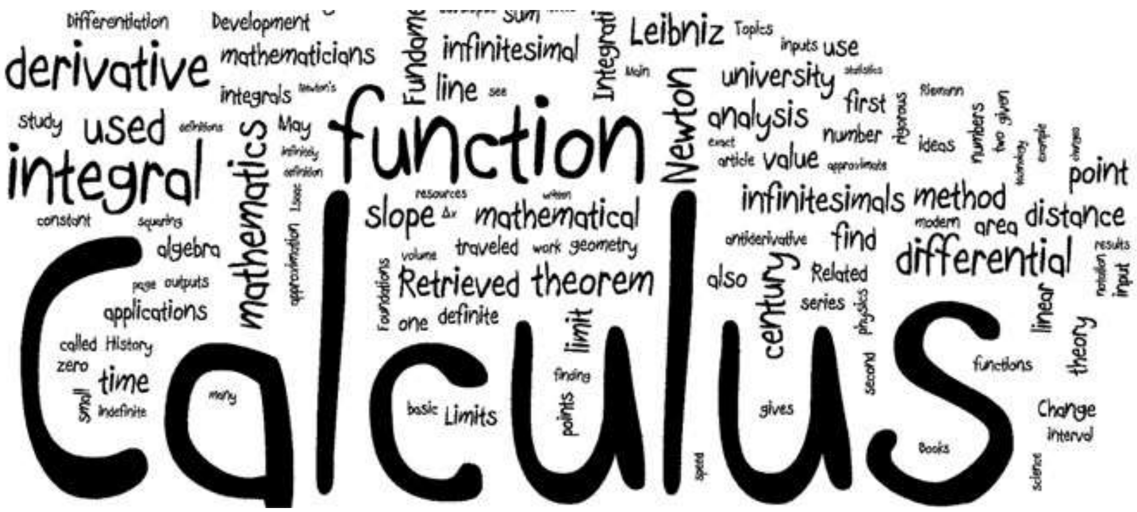


AP CALCULUS BC

STUFF YOU MUST KNOW COLD . . .

Alternate Definition of the Derivative:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Basic Derivatives

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos\ x$$

$$\frac{d}{dx}(\cos x) = -\sin\ x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$$

$$\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$$

Where u is a function of x,
and a is a constant.

Differentiation Rules

Chain Rule:

$$\frac{d}{dx}[f(u)] = f'(u)\frac{du}{dx} \ \ OR \ \ \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Product Rule:

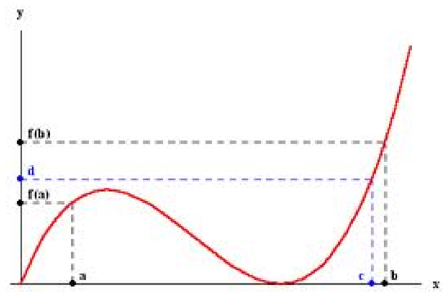
$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \ \ OR \ u\,v' + v\,u'$$

Quotient Rule:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \ \ OR \ \ \frac{v\,u' - u\,v'}{v^2}$$

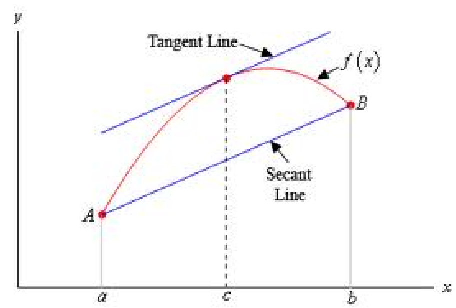
Intermediate Value Theorem

If the function $f(x)$ is continuous on $[a, b]$, and y is a number between $f(a)$ and $f(b)$, then there exists at least one number $x = c$ in the open interval (a, b) such that $f(c) = y$.



Mean Value Theorem

If the function $f(x)$ is continuous on $[a, b]$, **AND** *<u>the first derivative exists on the interval</u>* (a, b) then there is at least one number $x = c$ in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

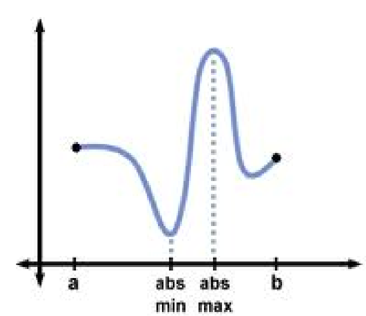


Rolle's Theorem

If the function $f(x)$ is continuous on $[a, b]$, **AND** *<u>the first derivative exists on the interval</u>* (a, b) **AND** $f(a) = f(b)$, then there is at least one number $x = c$ in (a, b) such that $f'(c) = 0$.

Extreme Value Theorem

If the function $f(x)$ is continuous on $[a, b]$, then the function is guaranteed to have an absolute maximum and an absolute minimum on the interval.



Derivative of an Inverse Function:

If f has an inverse function g then:

$$g'(x) = \frac{1}{f'(g(x))}$$

derivatives are reciprocal slopes

Implicit Differentiation

Remember that in implicit differentiation you will have a $\frac{dy}{dx}$ for each y in the original function or equation. Isolate the $\frac{dy}{dx}$. If you are taking the second derivative $\frac{d^2y}{dx^2}$, you will often substitute the expression you found for the first derivative somewhere in the process.

Average Rate of Change ARoC:

$$m_{sec} = \frac{f(b) - f(a)}{b - a}$$

Instantaneous Rate of Change IRoC:

$$m_{tan} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Curve Sketching And Analysis

$y = f(x)$ must be continuous at each:

Critical point: $\frac{dy}{dx} = 0$ or <u>undefined</u>

<u>LOOK OUT FOR ENDPOINTS</u>

Local minimum:

$\frac{dy}{dx}$ goes $(-, 0, +)$ or $(-, und, +)$ OR $\frac{d^2y}{dx^2} > 0$

Local maximum:

$\frac{dy}{dx}$ goes $(+, 0, -)$ or $(+, und, -)$ OR $\frac{d^2y}{dx^2} < 0$

Point of inflection: concavity changes

$\frac{d^2y}{dx^2}$ goes from $(+, 0, -)$, $(-, 0, +)$, $(+, und, -)$, OR $(-, und, +)$

First Derivative:

$f'(x) > 0$ function is increasing.

$f'(x) < 0$ function is decreasing.

$f'(x) = 0$ or DNE: Critical Values at x.

Relative Maximum: $f'(x) = 0$ or DNE and sign of $f'(x)$ changes from $+$ to $-$.

Relative Minimum: $f'(x) = 0$ or DNE and sign of $f'(x)$ changes from $-$ to $+$.

Absolute Max or Min:

<u>MUST CHECK ENDPOINTS ALSO</u>

The maximum value is a y-value.

Second Derivative:

$f''(x) > 0$ function is concave up.

$f''(x) < 0$ function is concave down.

$f'(x) = 0$ and sign of $f''(x)$ changes, then there is a point of inflection at x.

Relative Maximum: $f''(x) < 0$

Relative Minimum: $f''(x) > 0$

Write the equation of a tangent line at a point:

You need a slope (derivative) and a point.

$$y_2 - y_1 = m\,(x_2 - x_1)$$

Horizontal Asymptotes:

1. If the largest exponent in the numerator is < largest exponent in the denominator then $\lim_{x \to \pm\infty} f(x) = 0$.

2. If the largest exponent in the numerator is > the largest exponent in the denominator then $\lim_{x \to \pm\infty} f(x) = DNE$

3. If the largest exponent in the numerator is = to the largest exponent in the denominator then the quotient of the leading coefficients is the asymptote. $\lim_{x \to \pm\infty} f(x) = \frac{a}{b}$

ONLY FOUR THINGS YOU CAN DO ON A CALCULATOR THAT NEEDS NO WORK SHOWN:

1. Graphing a function within an arbitrary view window.
2. Finding the zeros of a function.
3. Computing the derivative of a function numerically.
4. Computing the definite integral of a function numerically.

Distance, Velocity, and Acceleration

$x(t)$ = position function
$v(t)$ = velocity function
$a(t)$ =acceleration function

The derivative of position (*ft*) is velocity (*ft/sec*); the derivative of velocity (*ft/sec*) is acceleration (*ft/sec*^{2}).

The integral of acceleration (*ft/sec*^{2}) is velocity (*ft/sec*) ; the integral of velocity (*ft/sec*) is position (*ft*).

Speed is | velocity |

If acceleration and velocity have the ***same sign***, then the speed is *increasing*, particle is moving right.

If the acceleration and velocity have ***different signs***, then the speed is *decreasing*, particle is moving left.

Displacement = $\int_{t_0}^{t_f} v(t)\, dt$

Distance $= \int_{initial\ time}^{final\ time} |\, v(t)|\ dt$

Average Velocity

$$= \frac{\text{final position} - \text{initial position}}{\text{total time}} = \frac{\Delta x}{\Delta t}$$

The Accumulation Function

$$F(x) = f(a) + \int_a^x f'(t)\, dt$$

The total amount, $F(x)$, at any time x , is the initial amount, $f\ (a)$, plus the amount of change between $t = a$ and $t = x$, given by the integral.

LOGARITHMS
Definition:

$$\ln N = p \leftrightarrow e^p = N$$

$\ln e = 1$
$\ln 1 = 0$
$\ln(MN) = \ln M + \ln N$
$\ln\left(\frac{M}{N}\right) = \ln M - \ln N$
$p \cdot \ln M = \ln M^p$

EXPONENTIAL GROWTH and DECAY:

When you see these words use: $y = Ce^{kt}$

"y is a differentiable function of t such that $y > 0$ *and* $y' = ky$ *"*

"the rate of change of y is proportional to y"

When solving a differential equation:

1. Separate variables first
2. Integrate
3. Add +C to one side
4. Use initial conditions to find "C"
5. Write the equation if the form of $y = f(x)$

"PLUS A CONSTANT"

The Fundamental Theorem of Calculus

$$\int_a^b f(x)dx = F(b) - F(a)$$

Where $F'(x) = f(x)$

Corollary to FTC

$$\frac{d}{dx}\int_a^{g(u)} f(t)dt = f\big(g(u)\big)\,\frac{du}{dx}$$

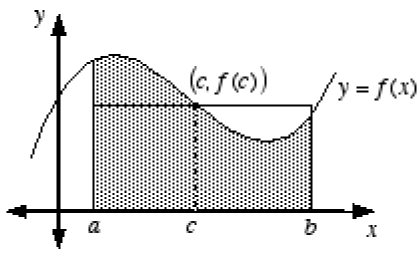
Mean Value Theorem for Integrals:
The Average Value

If the function $f(x)$ is continuous on $[a, b]$ and the first derivative exists on the interval (a, b), then there exists a number $x = c$ on (a, b) such that

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{\int_a^b f(x)dx}{b-a}$$

This value $f(c)$ is the "average value" of the function on the interval $[a, b]$.

The rectangle has the same area as the shaded region under the curve.



Riemann Sums

A Riemann Sum means a rectangular approximation. Approximation means that you ***DO NOT EVALUATE THE INTEGRAL***; you add up the areas of the rectangles.

Trapezoidal Rule

For uneven intervals, may need to calculate area of one trapezoid at a time and total.

$A_{Trap} = \frac{1}{2}h[b_1 + b_2]$

For even intervals:

$$\int_a^b f(x)\,dx = \frac{b-a}{2n}\left[\begin{matrix} y_0 + 2y_1 + 2y_2 + \ldots \\ +2y_{n-1} + y_n \end{matrix}\right]$$

Values of Trigonometric Functions for Common Angles

θ	$sin\ \theta$	$cos\ \theta$	$tan\ \theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	"∞"
π	0	-1	0

Must know both inverse trig and trig values:

EX. $tan\ \frac{\pi}{4} = 1$ and $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

ODD and EVEN:

$\sin(-x) = -\sin x$ (odd)

$\cos(-x) = \cos x$ (even)

Trigonometric Identities

Pythagorean Identities:

$sin^2\theta + cos^2\theta = 1$

The other two are easy to derive by dividing by $\sin^2\theta$ or $\cos^2\theta$.

$1 + \tan^2\theta = sec^2\theta$

$\cot^2\theta + 1 = \csc^2\theta$

Double Angle Formulas:

$\sin 2x = 2\sin x\cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$

Power-Reducing Formulas:

$\cos^2 x = \frac{1}{2}(1 + \cos 2x\ )$

$\sin^2 x = \frac{1}{2}\ (1 - \cos 2x)$

Quotient Identities:

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\quad\cot\theta = \frac{\cos\theta}{\sin\theta}$

Reciprocal Identities:

$csc\ x = \frac{1}{sin\ x}$ or $\sin x\csc x = 1$

$sec\ x = \frac{1}{\cos x}$ or $\cos x\sec x = 1$

Basic Integrals

$$\int du = u + C$$

$$\int u^n du = \frac{u^{n+1}}{n+1} + C \; n \neq -1$$

$$\int \frac{du}{u} = \ln|\,u\,| + C$$

$$\int e^u\, du = e^u + C$$

$$\int a^u\, du = \frac{a^u}{\ln a} + C$$

$$\int \sin u\, du = -\cos u + C$$

$$\int \cos u\, du = \sin u + C$$

$$\int \tan u\, du = -\ln|\cos u + C|$$

$$\int \cot u\, du = \ln|\sin u| + C$$

$$\int \sec u\, du = \ln|\sec u + \tan u| + C$$

$$\int \csc u\, du = -\ln|\csc u + \cot u| + C$$

$$\int \sec^2 u\, due = \tan u + C$$

$$\int \csc^2 u\, du = -\cot u + C$$

$$\int \sec u \tan u\, du = \sec u + C$$

$$\int \csc u \cot u\, du = -\csc u + C$$

Area and Solids of Revolution:

NOTE: $(\boldsymbol{a}, \boldsymbol{b})$ are *x*-coordinates and
$(\boldsymbol{c}, \boldsymbol{d})$ are *y*-coordinates

Area Between Two Curves:
Slices ⊥ to *x*-axis: $A = \int_a^b [f(x) - g(x)]\, dx$

Slices ⊥ to *y*-axis: $A = \int_c^d [f(y) - g(y)]\, dy$

Volume By Disk Method:

About *x*-axis: $V = \pi \int_a^b [R(x)]^2\, dx$

About *y*-axis: $V = \pi \int_c^d [R(y)]^2\, dy$

Volume By Washer Method:

About *x*-axis: $V = \pi \int_a^b ([R(x)]^2 - [r(x)]^2)\, dx$

About *y*-axis: $V = \pi \int_c^d ([R(y)]^2 - [r(y)]^2)\, dy$

Volume By Shell Method:

About *x*-axis: $V = 2\,\pi \int_c^d y\, [R(y)]\, dy$

About *y*-axis: $V = 2\,\pi \int_a^b x\, [R(x)]\, dx$

General Equations for Known Cross Section where *base* is the distance between the two curves and *a* and *b* are the limits of integration.

SQUARES: $V = \int_a^b (base)^2\, dx$

TRIANGLES

EQUILATERAL: $V = \frac{\sqrt{3}}{4} \int_a^b (base)^2\, dx$

ISOSCELES RIGHT: $V = \frac{1}{4} \int_a^b (base)^2\, dx$

RECTANGLES: $V = \int_a^b (base) \cdot h\, dx$
where h is the height of the rectangles.

SEMI-CIRCLES: $V = \frac{\pi}{2} \int_a^b (radius)^2\, dx$
where radius is ½ distance between the two curves.

MORE DERIVATIVES:

$$\frac{d}{dx}\left[\sin^{-1}\frac{u}{a}\right] = \frac{1}{\sqrt{a^2-u^2}}\,\frac{du}{dx}$$

$$\frac{d}{dx}[\cos^{-1}x] = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left[\tan^{-1}\frac{u}{a}\right] = \frac{a}{a^2+u^2}\,\frac{du}{dx}$$

$$\frac{d}{dx}[\cot^{-1}x] = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}\left[\sec^{-1}\frac{u}{a}\right] = \frac{a}{|u|\sqrt{u^2-a^2}}\,\frac{du}{dx}$$

$$\frac{d}{dx}[\csc^{-1}x] = \frac{-1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(a^u) = a^u \ln a\,\frac{du}{dx}$$

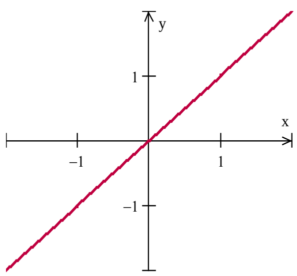
$$\frac{d}{dx}[\log_a x] = \frac{1}{x\ln a}$$

MORE INTEGRALS:

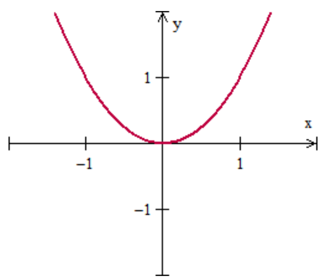
$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

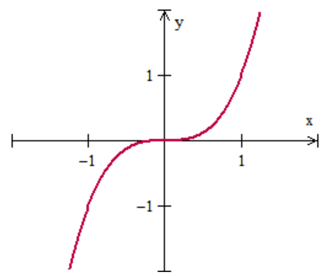
$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{|u|}{a} + C$$



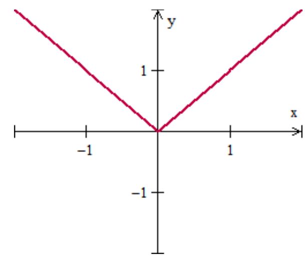
$y = x$



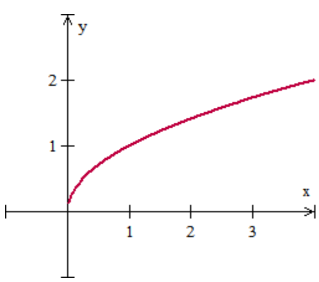
$y = x^2$



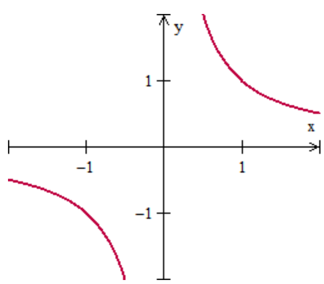
$y = x^3$



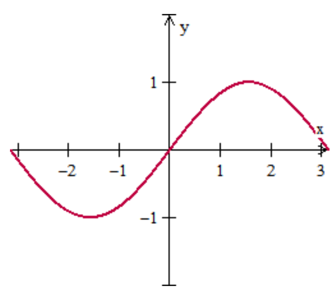
$y = |x|$



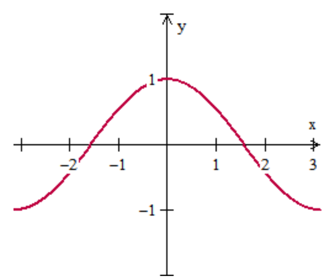
$y = \sqrt{x}$



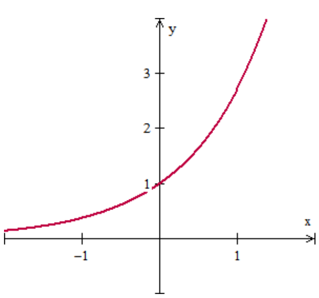
$y = \frac{1}{x}$



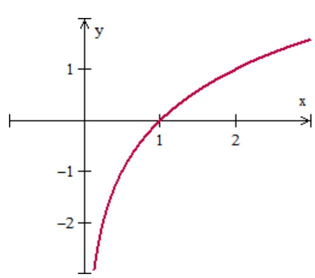
$y = \sin x$



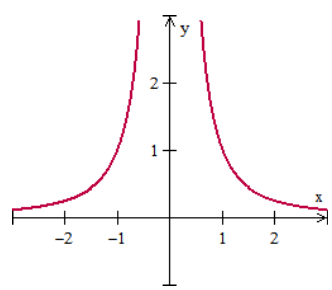
$y = \cos x$



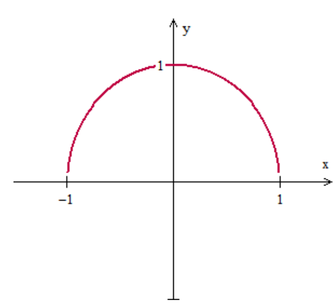
$y = e^x$



$y = \ln x$



$y = \frac{1}{x^2}$



$y = \sqrt{a^2 - x^2}$

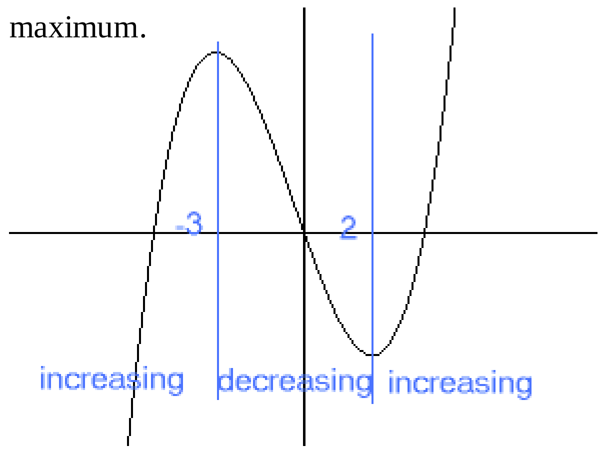
Extrema, Increasing/Decreasing Functions, the First Derivative Test and the Second Derivative Test

Finding Extrema on a Closed Interval [a,b]

1) Find the critical numbers of *f(x)*.
2) Evaluate *f(x)* at each critical number.
3) Evaluate *f(x)* at the endpoints.
4) The least value is a minimum. The greatest value is the maximum.

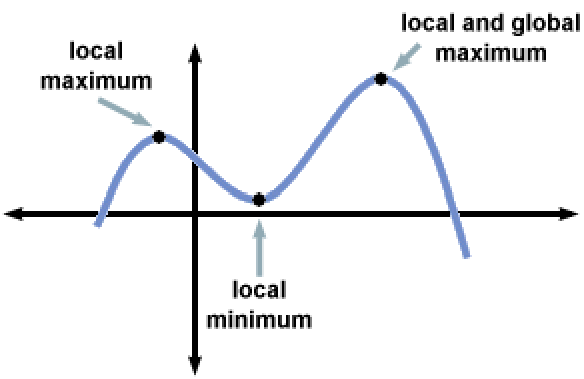
Determining if *f(x)* is Increasing or Decreasing on (a,b)

1) Find the critical numbers of *f(x)*.
2) Determine the intervals of *f(x)* to test.
3) Determine the sign of $f'(x)$ at one value in the intervals.
4) If $f'(x) > 0$, then *f(x)* is increasing on the interval (a,b).
5) If $f'(x) < 0$, then *f(x)* is decreasing on the interval (a,b).
6) If $f'(x) = 0$, then *f(x)* is constant on (a,b).



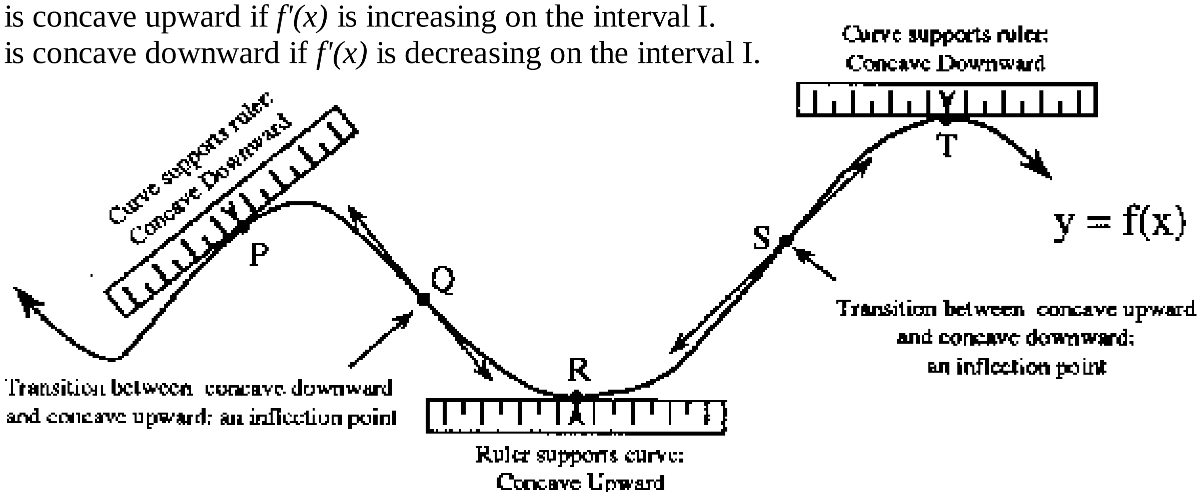
The First Derivative Test (c is a critical number of *f(x)*)

1) If $f'(x)$ changes from negative to positive at c, then *f(c)* is a relative (local) minimum of *f(x)*
2) If $f'(x)$ changes from positive to negative at c, then *f(c)* is a relative (local) maximum of *f(x)*.



Definition of Concavity

1) *f(x)* is concave upward if $f'(x)$ is increasing on the interval I.
2) *f(x)* is concave downward if $f'(x)$ is decreasing on the interval I.

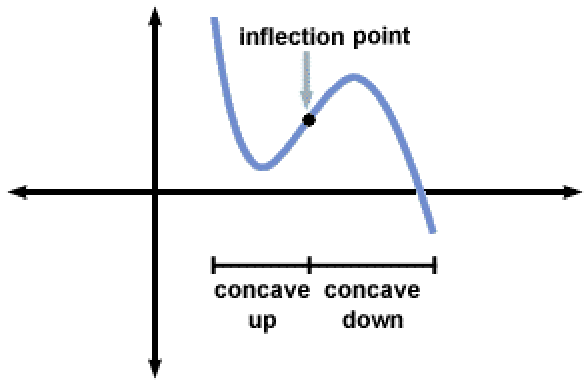


Determining if *f(x)* is Concave Up or Down

1) Find $f''(x)$ and locate the points at which $f''(x) = 0$ or is undefined.
2) Use the points found in #1 to determine your test intervals.
3) Evaluate one test point from each of your intervals.
4) If $f''(x) > 0$, then $f(x)$ is concave up on the interval.
5) If $f''(x) < 0$, then $f(x)$ is concave down on the interval.

Points of Inflection

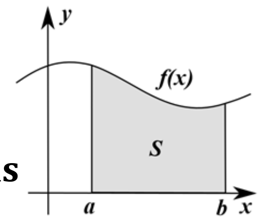
Points of inflection occur when the graph of $f(x)$ changes from concave up to concave down (or vice versa). Points of inflection only occur at values where $f''(x) = 0$ or is undefined. **NOTE: not all values of $f''(x) = 0$/undefined are points of inflection, therefore we must always check these points.**



Second Derivative Test (c is a critical number)

1) Find the critical numbers of f(x) {$f'(x) = 0$ or undefined}.
2) If $f''(c) > 0$, then $f(c)$ is a relative minimum because $f(c)$ is concave up.
3) If $f''(c) < 0$, then $f(c)$ is a relative maximum because $f(c)$ is concave down.
4) If $f''(c) = 0$, then the test fails. Use the first derivative test.

10 Things to know for the Free Response Questions (And Mrs. Berkson's Tests)



1. You will be given 6 Free Response questions. For two questions you are allowed to use the graphing calculator and for the remaining four there is no calculator allowed. Each Free Response Question is worth 9 points. Not all parts are weighted equally.
2. Always round to 4 decimal places. (AP only requires 3 but 4 will always get you points).
3. No simplification is needed; $e^0 - 4 + 6$ is okay! If you simply and you simplify wrong you will be awarded no points!
4. If you think it, write it. Never give a bald answer without any supporting work. If just the answer were okay then it would be a multiple-choice question, not free response.
5. Answer the question; don't say too much. If you say something correctly and then begin to say additional wrong information you will lose points.
6. Never erase. Graders are trained to ignore crossed out work.
7. Always bring the problem back to Calculus. Never use "it" or "the function" when justifying an answer. You must use the name of the function you are describing. Calculus always gives you the points. Pre-Calculus will sometimes give you the points.

   Ex. $f'(x)$ is positive (Calculus) vs.

   $f(x)$ is increasing (Pre-Calculus)
8. Don't use calculator syntax. If you use your calculator, describe it clearly in math terms, not in calculator terms.
9. Watch for linkage issues. Use arrows instead of equal signs.
10. Don't write $f(x) = 2(1.5) + 3$ when you mean $f(1.5) = 2(1.5) + 3$.

AP Calculus – Final Review Sheet

When you see the words ….	This is what you think of doing
1. Find the zeros	Set function = 0, factor or use quadratic equation if quadratic, graph to find zeros on calculator
2. Find equation of the line tangent to $f(x)$ on $[a,b]$	Take derivative - $f'(a)=m$ and use $y-y_1=m(x-x_1)$
3. Find equation of the line normal to $f(x)$ on $[a,b]$	Same as above but $m=\dfrac{-1}{f'(a)}$
4. Show that $f(x)$ is even	Show that $f(-x)=f(x)$ - symmetric to *y*-axis
5. Show that $f(x)$ is odd	Show that $f(-x)=-f(x)$ - symmetric to origin
6. Find the interval where $f(x)$ is increasing	Find $f'(x)$, set both numerator and denominator to zero to find critical points, make sign chart of $f'(x)$ and determine where it is positive.
7. Find interval where the slope of $f(x)$ is increasing	Find the derivative of $f'(x)=f''(x)$, set both numerator and denominator to zero to find critical points, make sign chart of $f''(x)$ and determine where it is positive.
8. Find the minimum value of a function	Make a sign chart of $f'(x)$, find all relative minimums and plug those values back into $f(x)$ and choose the smallest.
9. Find the minimum slope of a function	Make a sign chart of the derivative of $f'(x)=f''(x)$, find all relative minimums and plug those values back into $f'(x)$ and choose the smallest.
10. Find critical values	Express $f'(x)$ as a fraction and set both numerator and denominator equal to zero.
11. Find inflection points	Express $f''(x)$ as a fraction and set both numerator and denominator equal to zero. Make sign chart of $f''(x)$ to find where $f''(x)$ changes sign. (+ to – or – to +)
12. Show that $\lim\limits_{x\to a} f(x)$ exists	Show that $\lim\limits_{x\to a^-} f(x)=\lim\limits_{x\to a^+} f(x)$
13. Show that $f(x)$ is continuous	Show that 1) $\lim\limits_{x\to a} f(x)$ exists ($\lim\limits_{x\to a^-} f(x)=\lim\limits_{x\to a^+} f(x)$) 2) $f(a)$ exists 3) $\lim\limits_{x\to a} f(x)=f(a)$
14. Find vertical asymptotes of $f(x)$	Do all factor/cancel of $f(x)$ and set denominator = 0
15. Find horizontal asymptotes of $f(x)$	Find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$
16. Find the average rate of change of $f(x)$ on $[a,b]$	Find $\dfrac{f(b)-f(a)}{b-a}$
17. Find instantaneous rate of change of $f(x)$ at a	Find $f'(a)$

18. Find the average value of $f(x)$ on $[a,b]$	Find $\dfrac{\int_a^b f(x)\,dx}{b-a}$
19. Find the absolute maximum of $f(x)$ on $[a,b]$	Make a sign chart of $f'(x)$, find all relative maximums and plug those values back into $f(x)$ as well as finding $f(a)$ and $f(b)$ and choose the largest.
20. Show that a piecewise function is differentiable at the point *a* where the function rule splits	First, be sure that the function is continuous at $x=a$. Take the derivative of each piece and show that $\lim_{x\to a^-} f'(x)=\lim_{x\to a+} f'(x)$
21. Given $s(t)$ (position function), find $v(t)$	Find $v(t)=s'(t)$
22. Given $v(t)$, find how far a particle travels on $[a,b]$	Find $\int_a^b \lvert v(t)\rvert dt$
23. Find the average velocity of a particle on $[a,b]$	Find $\dfrac{\int_a^b v(t)dt}{b-a}=\dfrac{s(b)-s(a)}{b-a}$
24. Given $v(t)$, determine if a particle is speeding up at $t=k$	Find $v(k)$ and $a(k)$. Multiply their signs. If both positive, the particle is speeding up, if different signs, then the particle is slowing down.
25. Given $v(t)$ and $s(0)$, find $s(t)$	$s(t)=\int v(t)dt+C$ Plug in $t=0$ to find C
26. Show that Rolle's Theorem holds on $[a,b]$	Show that *f* is continuous and differentiable on the interval. If $f(a)=f(b)$, then find some *c* in $[a,b]$ such that $f'(c)=0$.
27. Show that Mean Value Theorem holds on $[a,b]$	Show that *f* is continuous and differentiable on the interval. Then find some *c* such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.
28. Find domain of $f(x)$	Assume domain is $(-\infty,\infty)$. Restrictable domains: denominators $\neq 0$, square roots of only non negative numbers, log or ln of only positive numbers.
29. Find range of $f(x)$ on $[a,b]$	Use max/min techniques to rind relative max/mins. Then examine $f(a), f(b)$
30. Find range of $f(x)$ on $(-\infty,\infty)$	Use max/min techniques to rind relative max/mins. Then examine $\lim_{x\to\pm\infty} f(x)$.
31. Find $f'(x)$ by definition	$f'(x)=\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ or $f'(x)=\lim_{x\to a}\dfrac{f(x)-f(a)}{x-a}$
32. Find derivative of inverse to $f(x)$ at $x=a$	Interchange *x* with *y*. Solve for $\dfrac{dy}{dx}$ implicitly (in terms of *y*). Plug your *x* value into the inverse relation and solve for *y*. Finally, plug that *y* into your $\dfrac{dy}{dx}$.

33. y is increasing proportionally to y	$\frac{dy}{dt} = ky$ translating to $y = Ce^{kt}$
34. Find the line $x = c$ that divides the area under $f(x)$ on $[a,b]$ to two equal areas	$\int_a^c f(x)dx = \int_c^b f(x)dx$
35. $\frac{d}{dx}\int_a^x f(t)\,dt =$	2nd FTC: Answer is $f(x)$
36. $\frac{d}{dx}\int_a^u f(t)dt$	2nd FTC: Answer is $f(u)\frac{du}{dx}$
37. The rate of change of population is …	$\frac{dP}{dt} = \ldots$
38. The line $y = mx + b$ is tangent to $f(x)$ at (x_1, y_1)	Two relationships are true. The two functions share the same slope ($m = f'(x)$) and share the same *y* value at x_1.
39. Find area using left Riemann sums	$A = base[x_0 + x_1 + x_2 + \ldots + x_{n-1}]$
40. Find area using right Riemann sums	$A = base[x_1 + x_2 + x_3 + \ldots + x_n]$
41. Find area using midpoint rectangles	Typically done with a table of values. Be sure to use only values that are given. If you are given 6 sets of points, you can only do 3 midpoint rectangles.
42. Find area using trapezoids	$A = \frac{base}{2}[x_0 + 2x_1 + 2x_2 + \ldots + 2x_{n-1} + x_n]$ This formula only works when the base is the same. If not, you have to do individual trapezoids.
43. Solve the differential equation …	Separate the variables – *x* on one side, *y* on the other. The *dx* and *dy* must all be upstairs.
44. Meaning of $\int_a^x f(t)\,dt$	The accumulation function – accumulated area under the function $f(x)$ starting at some constant *a* and ending at *x*.
45. Given a base, cross sections perpendicular to the *x*-axis are squares	The area between the curves typically is the base of your square. So the volume is $\int_a^b (base^2)dx$
46. Find where the tangent line to $f(x)$ is horizontal	Write $f'(x)$ as a fraction. Set the numerator equal to zero.
47. Find where the tangent line to $f(x)$ is vertical	Write $f'(x)$ as a fraction. Set the denominator equal to zero.
48. Find the minimum acceleration given $v(t)$	First find the acceleration $a(t) = v'(t)$. Then minimize the acceleration by examining $a'(t)$.
49. Approximate the value of $f(0.1)$ by using the tangent line to *f* at $x = 0$	Find the equation of the tangent line to *f* using $y - y_1 = m(x - x_1)$ where $m = f'(0)$ and the point is $(0, f(0))$. Then plug in 0.1 into this line being sure to use an approximate $(\approx)$ sign.

50. Given the value of $F(a)$ and the fact that the anti-derivative of *f* is *F*, find $F(b)$1	Usually, this problem contains an antiderivative you cannot take. Utilize the fact that if $F(x)$ is the antiderivative of *f*, then $\int_a^b F(x)dx = F(b) - F(a)$. So solve for $F(b)$ using the calculator to find the definite integral.
51. Find the derivative of $f(g(x))$	$f'(g(x)) \cdot g'(x)$
52. Given $\int_a^b f(x)\,dx$, find $\int_a^b [f(x)+k]\,dx$	$\int_a^b [f(x)+k]\,dx = \int_a^b f(x)dx + \int_a^b kdx$
53. Given a picture of $f'(x)$, find where $f(x)$ is increasing	Make a sign chart of $f'(x)$ and determine where $f'(x)$ is positive.
54. Given $v(t)$ and $s(0)$, find the greatest distance from the origin of a particle on $[a,b]$	Generate a sign chart of $v(t)$ to find turning points. Then integrate $v(t)$ using $s(0)$ to find the constant to find $s(t)$. Finally, find *s*(all turning points) which will give you the distance from your starting point. Adjust for the origin.
55. Given a water tank with *g* gallons initially being filled at the rate of $F(t)$ gallons/min and emptied at the rate of $E(t)$ gallons/min on $[t_1, t_2]$, find a) the amount of water in the tank at *m* minutes	$g + \int_t^{t_2} (F(t) - E(t))dt$
56. b) the rate the water amount is changing at *m*	$\frac{d}{dt}\int_t^m (F(t) - E(t))dt = F(m) - E(m)$
57. c) the time when the water is at a minimum	$F(m) - E(m)$=0, testing the endpoints as well.
58. Given a chart of *x* and $f(x)$ on selected values between *a* and *b*, estimate $f'(c)$ where *c* is between *a* and b.	Straddle *c*, using a value *k* greater than *c* and a value *h* less than *c*. so $f'(c) \approx \frac{f(k) - f(h)}{k - h}$
59. Given $\frac{dy}{dx}$, draw a slope field	Use the given points and plug them into $\frac{dy}{dx}$, drawing little lines with the indicated slopes at the points.
60. Find the area between curves $f(x), g(x)$ on $[a,b]$	$A = \int_a^b [f(x) - g(x)]dx$, assuming that the *f* curve is above the *g* curve.
61. Find the volume if the area between $f(x), g(x)$ is rotated about the *x*-axis	$V = \pi\int_a^b \left[(f(x))^2 - (g(x))^2 \right] dx$ assuming that the *f* curve is above the *g* curve.

BC Problems

62. Find $\lim_{x\to\infty}\frac{f(x)}{g(x)}$ if $\lim_{x\to\infty} f(x)=\lim_{x\to\infty} g(x)=0$	Use L'Hopital's Rule.
63. Find $\int_0^{\infty} f(x)\ dx$	$\lim_{h\to\infty}\int_0^h f(x)dx$
64. $\frac{dP}{dt}=\frac{k}{M}P(M-P)$ or $\frac{dP}{dt}=kP\left(1-\frac{P}{M}\right)$	Signals logistic growth. $\lim_{t\to\infty}\frac{dP}{dt}=0\Rightarrow M=P$
65. Find $\int\frac{dx}{x^2+ax+b}$ where x^2+ax+b factors	Factor denominator and use Heaviside partial fraction technique.
66. The position vector of a particle moving in the plane is $r(t)=\langle x(t),y(t)\rangle$ a) Find the velocity.	$v(t)=\langle x'(t),y'(t)\rangle$
67. b) Find the acceleration.	$a(t)=\langle x''(t),y''(t)\rangle$
68. c) Find the speed.	$\lVert v(t)\rVert=\sqrt{[x'(t)]^2+[y'(t)]^2}$
69. a) Given the velocity vector $v(t)=\langle x(t),y(t)\rangle$ and position at time 0, find the position vector.	$s(t)=\int x(t)dt+\int y(t)dt+C$ Use $s(0)$ to find *C*, remembering it is a vector.
70. b) When does the particle stop?	$v(t)=0\to x(t)=0$ AND $y(t)=0$
71. c) Find the slope of the tangent line to the vector at t_1.	This is the acceleration vector at t_1.
72. Find the area inside the polar curve $r=f(\theta)$.	$A=\frac{1}{2}\int_{\theta_1}^{\theta_2}[f(\theta)]^2d\theta$
73. Find the slope of the tangent line to the polar curve $r=f(\theta)$.	$x=r\cos\theta, y=r\sin\theta\Rightarrow\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$
74. Use Euler's method to approximate $f(1.2)$ given $\frac{dy}{dx}$, $(x_0,y_0)=(1,1)$, and $\Delta x=0.1$	$dy=\frac{dy}{dx}(\Delta x), y_{\text{new}}=y_{\text{old}}+dy$
75. Is the Euler's approximation an underestimate or an overestimate?	Look at sign of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the interval. This gives you increasing.decreasing/concavity. Draw picture to ascertain

	answer.
76. Find $\int x^n e^{ax} dx$ where a, n are integers	Integration by parts, $\int u\,dv = uv - \int v\,du + C$
77. Write a series for $x^n \cos x$ where n is an integer	$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$ Multiply each term by x^n
78. Write a series for $\ln(1+x)$ centered at $x=0$.	Find Maclaurin polynomial: $P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n$
79. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if…..	$p > 1$
80. If $f(x) = 2 + 6x + 18x^2 + 54x^3 + \ldots$, find $f\left(-\frac{1}{2}\right)$	Plug in and factor. This will be a geometric series: $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$
81. Find the interval of convergence of a series.	Use a test (usually the ratio) to find the interval and then test convergence at the endpoints.
82. Let S_4 be the sum of the first 4 terms of an alternating series for $f(x)$. Approximate $\lvert f(x) - S_4 \rvert$	This is the error for the 4th term of an alternating series which is simply the 5th term. It will be positive since you are looking for an absolute value.
83. Suppose $f^{(n)}(x) = \frac{(n+1)\ n!}{2^n}$. Write the first four terms and the general term of a series for $f(x)$ centered at $x = c$	You are being given a formula for the derivative of $f(x)$. $f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n$
84. Given a Taylor series, find the Lagrange form of the remainder for the n^{th} term where n is an integer at $x = c$.	You need to determine the largest value of the 5th derivative of f at some value of z. Usually you are told this. Then: $R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$
85. $f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$	$f(x) = e^x$
86. $f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \ldots$	$f(x) = \sin x$
87. $f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + \frac{(-1)^n x^{2n}}{(2n)!} + \ldots$	$f(x) = \cos x$
88. Find $\int (\sin x)^m (\cos x)^n dx$ where m and n are integers	If m is odd and positive, save a sine and convert everything else to cosine. The sine will be the *du*. If n is odd and positive, save a cosine and convert everything else to sine. The cosine will be the *du*. Otherwise use the fact that:

	$\sin^2 x = \frac{1-\cos 2x}{2}$ and $\cos^2 x = \frac{1+\cos 2x}{2}$
89. Given $x = f(t), y = g(t)$, find $\frac{dy}{dx}$	$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$
90. Given $x = f(t), y = g(t)$, find $\frac{d^2y}{dx^2}$	$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$
91. Given $f(x)$, find arc length on $[a,b]$	$L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$
92. $x = f(t), y = g(t)$, find arc length on $[t_1, t_2]$	$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$
93. Find horizontal tangents to a polar curve $r = f(\theta)$	$x = r\cos\theta, y = r\sin\theta$ Find where $r\sin\theta = 0$ where $r\cos\theta \neq 0$
94. Find vertical tangents to a polar curve $r = f(\theta)$	$x = r\cos\theta, y = r\sin\theta$ Find where $r\cos\theta = 0$ where $r\sin\theta \neq 0$
95. Find the volume when the area between $y = f(x), x = 0, y = 0$ is rotated about the y-axis.	Shell method: $V = 2\pi \int_0^b$ radius $\cdot$ height dx where b is the root.
96. Given a set of points, estimate the volume under the curve using Simpson's rule on $[a,b]$.	$A \approx \frac{b-a}{3n}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + ... + 4y_{n-1} + y_n\right]$
97. Find the dot product: $\langle u_1, u_2\rangle \cdot \langle v_1, v_2\rangle$	$\langle u_1, u_2\rangle \cdot \langle v_1, v_2\rangle = u_1v_1 + u_2v_2$
98. Multiply two vectors:	You get a scalar.

2.1 Rates of Change and Limits

Objectives:
- •Calculate average and instantaneous speeds
- •Define and calculate limits for function values and apply the properties of limits
- •Use the Sandwich Theorem to find certain limits indirectly.

Suppose you drive 200 miles, and it takes you 4 hours.

$$\text{average speed} = \frac{\text{distance}}{\text{elapsed time}} = \frac{\Delta x}{\Delta t}$$

If you look at your speedometer during this trip, it might read 65 mph. This is your instantaneous speed.

→

A rock falls from a high cliff.

The position of the rock is given by: $y = 16t^2$

After 2 seconds:

average speed:

What is the instantaneous speed at 2 seconds?

→

$$V_{\text{instantaneous}} = \frac{\Delta y}{\Delta t} = \frac{16(2+h)^2 - 16(2)^2}{h}$$

for some very small change in t

where h = some very small change in t

We can use the TI-Nspire to evaluate this expression for smaller and smaller values of h.

→

$$V_{\text{instantaneous}} = \frac{\Delta y}{\Delta t} = \frac{16(2+h)^2 - 16(2)^2}{h}$$

We can see that the velocity approaches 64 ft/sec as h becomes very small.

We say that the velocity has a limiting value of 64 as h approaches zero.

(Note that h never actually becomes zero.)

h	$\frac{\Delta y}{\Delta t}$
1	80
0.1	65.6
.01	64.16
.001	64.016
.0001	64.0016
.00001	64.0002

→

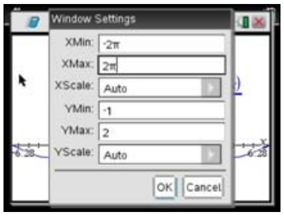
The limit as h approaches zero: $\displaystyle \lim_{h\to 0} \frac{16(2+h)^2 - 64}{h}$

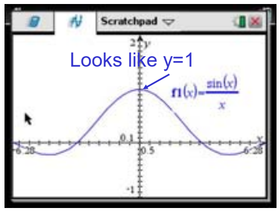
→

Consider: $y = \dfrac{\sin x}{x}$

What happens as x approaches zero?

Graphically:





→

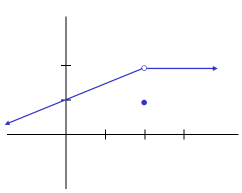
Limit notation: $\lim\limits_{x \to c} f(x) = L$

"The limit of f of x as x approaches c is L."

So: $\boxed{\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1}$

→

The <u>limit</u> of a function refers to the value that the function <u>approaches</u>, <u>not</u> the actual value (if any).



$\lim\limits_{x \to 2} f(x) =$

→

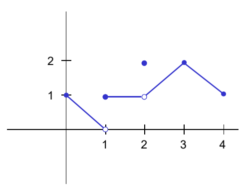
Properties of Limits:

Limits can be added, subtracted, multiplied, multiplied by a constant, divided, and raised to a power.

(See page 58 for details.)

For a limit to exist, the function must approach the same value from both sides.

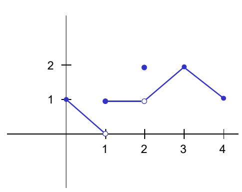
One-sided limits approach from either the left or right side only.



At x=1: $\lim\limits_{x \to 1^-} f(x) =$ ← left hand limit

$\lim\limits_{x \to 1^+} f(x) =$ ← right hand limit

$f(1) =$ ← value of the function



At x=2: $\lim\limits_{x \to 2^-} f(x) =$

$\lim\limits_{x \to 2^+} f(x) =$

$f(2) =$

At x=3: $\lim\limits_{x \to 3^-} f(x) =$

$\lim\limits_{x \to 3^+} f(x) =$

$f(3) =$

"Step functions" are sometimes used to describe real-life situations.

Our book refers to one such function: $y = \text{int}(x)$

This is the **Greatest Integer Function**.

The TI-89 contains the command $\text{int}(x)$, but it is important that you understand the function rather than just entering it in your calculator.

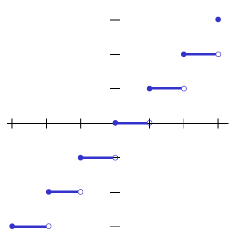
Greatest Integer Function:

$y = \text{greatest integer that is } \leq \text{x}$

The greatest integer function is also called the floor function.

The notation for the floor function is:

$y = \lfloor x \rfloor$



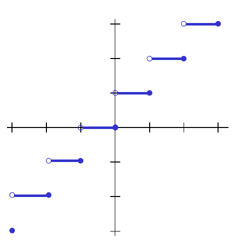
Some books use $y = [x]$ or $y = [\![x]\!]$.

We will not use these notations.

Least Integer Function:

$y = \text{least integer that is } \geq \text{x}$

x	y
0	0
0.5	1
0.75	1
1	1
1.5	2
2	2



Least Integer Function:

$y =$ least integer that is $\geq x$

The least integer function is also called the ceiling function.

The notation for the ceiling function is:

$$y = \lceil x \rceil$$

Don't worry, there are not wall functions, front door functions, fireplace functions!

→

The Sandwich Theorem:

If $g(x) \leq f(x) \leq h(x)$ for all $x \neq c$ in some interval about c and $\lim\limits_{x \to c} g(x) = \lim\limits_{x \to c} h(x) = L$, then $\lim\limits_{x \to c} f(x) = L$.

Show that: $\lim\limits_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$

The maximum value of sine is 1, so $x^2 \sin\left(\frac{1}{x}\right) \leq x^2$

The minimum value of sine is -1, so $x^2 \sin\left(\frac{1}{x}\right) \geq -x^2$

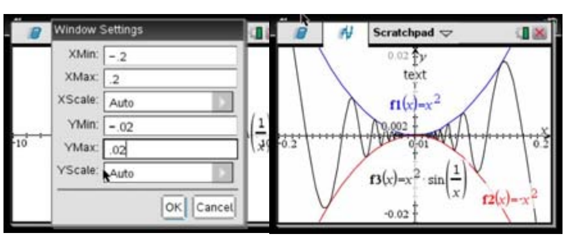
So: $-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$

→

$$\lim_{x \to 0} -x^2 \leq \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) \leq \lim_{x \to 0} x^2$$

$$0 \leq \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) \leq 0$$

By the sandwich theorem: $\lim\limits_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$

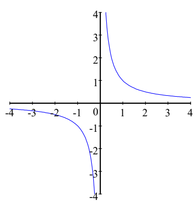


2.2 Limits Involving Infinity

Objectives:
- •Find and verify end behavior models for various functions
- •Calculate limits as $x \to \pm\infty$ and to identify vertical and horizontal asymptotes.

$$f(x) = \frac{1}{x}$$

$$\lim_{x\to\infty} \frac{1}{x} = 0$$



As the denominator gets larger, the value of the fraction gets smaller.

There is a horizontal asymptote if:

$\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$

→

Example 1:

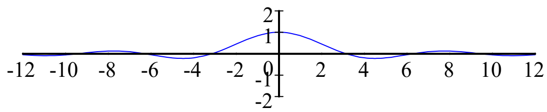
$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2}} = \lim_{x\to\infty} \frac{x}{x} = 1$$

This number becomes insignificant as $x \to \infty$.

$\therefore$ There is a horizontal asymptote at 1.

→

Example 2: $f(x) = \dfrac{\sin x}{x}$ Find: $\displaystyle\lim_{x\to\infty} \frac{\sin x}{x}$



$-1 \le \sin x \le 1$

so for $x > 0$: $\dfrac{-1}{x} \le \dfrac{\sin x}{x} \le \dfrac{1}{x}$

$$\lim_{x\to\infty} \frac{-1}{x} \le \lim_{x\to\infty} \frac{\sin x}{x} \le \lim_{x\to\infty} \frac{1}{x}$$

$$0 \le \lim_{x\to\infty} \frac{\sin x}{x} \le 0$$

$\therefore$ by the sandwich theorem:

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0$$

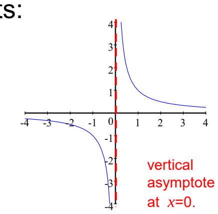
→

Example 3: Find: $\displaystyle\lim_{x\to\infty} \frac{5x + \sin x}{x}$

→

Infinite Limits:

$f(x) = \dfrac{1}{x}$



As the denominator approaches zero, the value of the fraction gets very large.

If the denominator is positive then the fraction is positive. $\displaystyle\lim_{x\to 0^+} \frac{1}{x} = \infty$

If the denominator is negative then the fraction is negative. $\displaystyle\lim_{x\to 0^-} \frac{1}{x} = -\infty$

→

Example 4:

$$\lim_{x\to 0^+} \frac{1}{x^2} = \infty$$

$$\lim_{x\to 0^-} \frac{1}{x^2} = \infty$$

The denominator is positive in both cases, so the limit is the same.

$$\therefore \lim_{x\to 0} \frac{1}{x^2} = \infty$$

→

End Behavior Models:

End behavior models model the behavior of a function as x approaches infinity or negative infinity.

A function g is:

a right end behavior model for f if and only if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$

a left end behavior model for f if and only if $\lim_{x\to-\infty} \frac{f(x)}{g(x)} = 1$

→

Example 7: $f(x) = x + e^{-x}$

As $x \to \infty$, e^{-x} approaches zero. (The x term dominates.)

$\therefore\ g(x) = x$ becomes a right-end behavior model.

$$\lim_{x\to\infty} \frac{x+e^{-x}}{x} = \lim_{x\to\infty} 1 + \frac{e^{-x}}{x} = 1 + 0 = 1$$

As $x \to -\infty$, e^{-x} increases faster than x decreases, therefore e^{-x} is dominant.

$\therefore\ h(x) = e^{-x}$ becomes a left-end behavior model.

Test of model $\lim_{x\to-\infty} \frac{x+e^{-x}}{e^{-x}} = \lim_{x\to-\infty} \frac{x}{e^{-x}} + 1 = 0 + 1 = 1$ Our model is correct.

→

Example 7: $f(x) = x + e^{-x}$

$\therefore\ g(x) = x$ becomes a right-end behavior model.

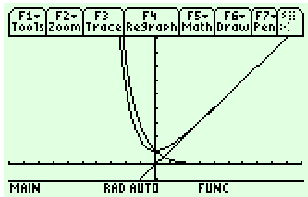
$\therefore\ h(x) = e^{-x}$ becomes a left-end behavior model.

On your calculator, graph:

$y_1 = x$

$y_2 = e^{-x}$

$y_3 = x + e^{-x}$



Use: $-10 \le x \le 10$

$-1 \le y \le 9$

→

Example 7:

$$f(x) = \frac{2x^5 + x^4 - x^2 + 1}{3x^2 - 5x + 7}$$

→

Often you can just “think through” limits.

$$\lim_{x \to \infty} \sin\left(\frac{1}{x}\right)$$

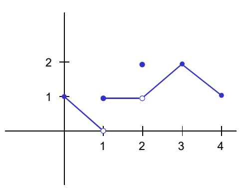
π

2.3 Continuity

Objectives:
- Identify the intervals upon which a given function is continuous and understand the meaning of a continuous function.
- Remove discontinuities by extending or modifying a function.
- Apply the Intermediate Value Theorem and the properties of algebraic combinations and composites of continuous functions.

Most of the techniques of calculus require that functions be <u>continuous</u>. A function is continuous if you can draw it in one motion without picking up your pencil.

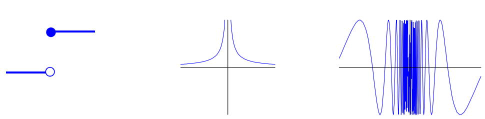
A function is continuous at a point if the limit is the same as the value of the function.



Removable Discontinuities:

(You can fill the hole.)

Essential Discontinuities:



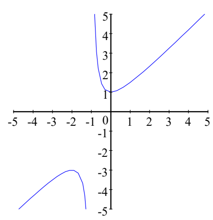
jump infinite oscillating

Removing a discontinuity:

$$f(x)=\frac{x^3-1}{x^2-1}$$

$$\lim_{x\to 1}\frac{x^3-1}{x^2-1}$$

Removing a discontinuity:



$$f(x)=\begin{cases}\dfrac{x^3-1}{x^2-1}, & x\neq 1\\[4pt] \dfrac{3}{2}, & x=1\end{cases}$$

Note: There is another discontinuity at $x=-1$ that can not be removed.

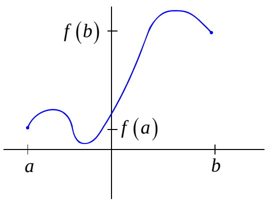
Continuous functions can be added, subtracted, multiplied, divided and multiplied by a constant, and the new function remains continuous.

Also: Composites of continuous functions are continuous.

examples: $y=\sin\left(x^2\right)$ $\quad$ $y=\left|\cos x\right|$

Intermediate Value Theorem

If a function is continuous between a and b, then it takes on every value between $f(a)$ and $f(b)$.



Because the function is continuous, it must take on every y value between $f(a)$ and $f(b)$.

→

Example 5: Is any real number exactly one less than its cube?

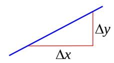
(Note that this doesn't ask what the number is, only if it exists.)

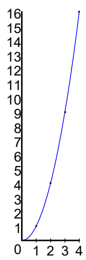
→

2.4 Rates of Change and Tangents Lines

Objectives:
- Apply directly the definition of slope of a curve in order to calculate slopes.
- Find the equations of the tangent line and normal line to a curve at a given point.
- Find the average rate of change of a function.

The slope of a line is given by: $m = \frac{\Delta y}{\Delta x}$





$f(x) = x^2$

The slope at (1,1) can be approximated by the slope of the secant through (4,16).

$\frac{\Delta y}{\Delta x}$

We could get a better approximation if we move the point closer to (1,1). ie: (3,9)

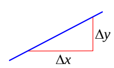
$\frac{\Delta y}{\Delta x}$

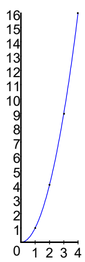
Even better would be the point (2,4).

$\frac{\Delta y}{\Delta x}$

→

The slope of a line is given by: $m = \frac{\Delta y}{\Delta x}$





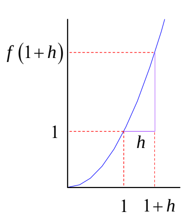
$f(x) = x^2$

If we got really close to (1,1), say (1.1,1.21), the approximation would get better still

$\frac{\Delta y}{\Delta x}$

How far can we go?

→



slope $= \frac{\Delta y}{\Delta x}$

slope at $(1,1)$

The slope of the curve $y = f(x)$ at the point $P(a, f(a))$ is:

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

→

The slope of the curve $y = f(x)$ at the point $P(a, f(a))$ is:

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$\dfrac{f(a+h) - f(a)}{h}$ is called the difference quotient of f at a.

If you are asked to find the slope using the definition or using the difference quotient, this is the technique you will use.

→

The slope of a curve at a point is the same as the slope of the tangent line at that point.

In the previous example, the tangent line could be found using $y - y_1 = m(x - x_1)$.

If you want the normal line, use the negative reciprocal of the slope. (in this case, $-\frac{1}{2}$)

(The normal line is perpendicular.)

→

Example 4:

Let $f(x) = \dfrac{1}{x}$

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

a) Find the slope at $x = a$.

→

Example 4:

Let $f(x) = \frac{1}{x}$

b) Where is the slope $-\frac{1}{4}$?

→

Review:

average slope: $m = \frac{\Delta y}{\Delta x}$

slope at a point: $m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

These are often mixed up by Calculus students!

average velocity: $V_{\text{ave}} = \frac{\text{total distance}}{\text{total time}}$

So are these!

instantaneous velocity: If $f(t)$ is the position function:

velocity = slope

$V = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$

π

3.1 Derivative of a Function

Objectives:
- Calculate slopes and derivatives using the definition of the derivative.
- Graph *f* from the graph of *f ′*, graph *f ′* from the graph of *f*, and graph the derivative of a function given numerically with data.

$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ is called the derivative of f at a.

We write: $f'(x)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

"The derivative of f with respect to x is …"

See pg. 99 and 100 for alternate definitions of derivatives.

There are many ways to write the derivative of $y=f(x)$

→

$f'(x)$ "f prime x" or "the derivative of f with respect to x"

y' "y prime"

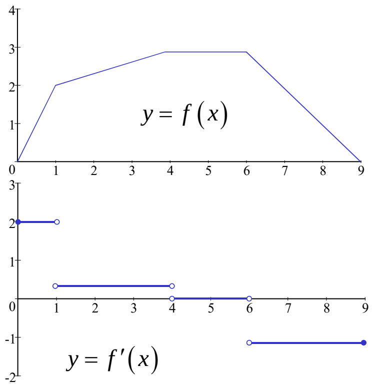
$\frac{dy}{dx}$ "dee why dee ecks" or "the derivative of y with respect to x"

$\frac{df}{dx}$ "dee eff dee ecks" or "the derivative of f with respect to x"

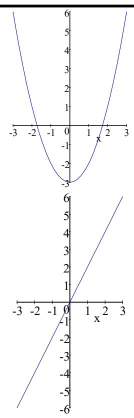
$\frac{d}{dx}f(x)$ "dee dee ecks uv eff uv ecks" or "the derivative of f of x"

(d dx of f of x)

See pg. 101 for uses of each notation



The derivative is the slope of the original function.



$$y = x^2 - 3$$

$$y' = \lim_{h \to 0} \frac{(x+h)^2 - 3 - (x^2 - 3)}{h}$$

$$y' = \lim_{h \to 0} \frac{\cancel{x^2} + 2x\cancel{h} + h^{\cancel{2}} - \cancel{x^2}}{\cancel{h}}$$

$$y' = \lim_{h \to 0} 2x + \cancelto{0}{h}$$

$$y' = 2x$$

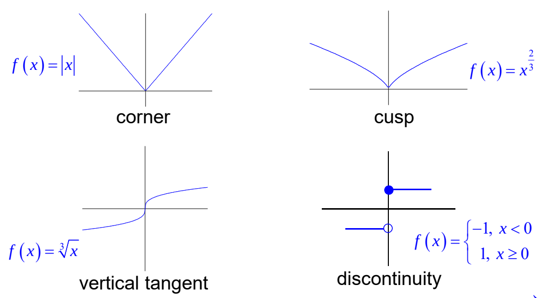
A function is <u>differentiable</u> if it has a derivative everywhere in its domain. It must be <u>continuous</u> and <u>smooth</u>. Functions on closed intervals must have one-sided derivatives defined at the end points.

3.2 Differentiability

Objectives:
- Find where a function is not differentiable and distinguish between corners, cusps, discontinuities, and vertical tangents
- Approximate derivatives numerically and graphically.

To be differentiable, a function must be continuous and smooth.

Derivatives will fail to exist at:



→

There are two theorems on page 113:

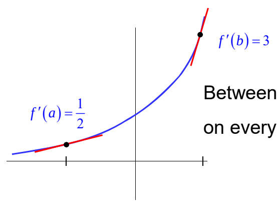
If f has a derivative at $x=a$, then f is continuous at $x=a$.

Since a function must be continuous to have a derivative, if it has a derivative then it is continuous.

→

Intermediate Value Theorem for Derivatives

If a and b are any two points in an interval on which f is differentiable, then f' takes on every value between $f'(a)$ and $f'(b)$.



Between a and b, f' must take on every value between $\frac{1}{2}$ and 3.

3.3 Rules for Differentiation

Objectives:
- Use the rules of differentiation to calculate derivatives, including second and higher order derivatives

If the derivative of a function is its slope, then for a constant function, the derivative must be zero.

$$\frac{d}{dx}(c)=0$$

example: $y=3$

$y'=0$

The derivative of a constant is zero.

We saw that if $y = x^2$, $y' = 2x$.

This is part of a pattern.

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

power rule

examples:

$f(x) = x^4$ $\qquad$ $y = x^8$

$f'(x) = 4x^3$ $\qquad$ $y' = 8x^7$

→

constant multiple rule:

$$\frac{d}{dx}(cu) = c\frac{du}{dx}$$

examples:

$$\frac{d}{dx}cx^n = cnx^{n-1}$$

$$\frac{d}{dx}7x^5 = 7\cdot 5x^4 = 35x^4$$

→

constant multiple rule:

$$\frac{d}{dx}(cu) = c\frac{du}{dx}$$

sum and difference rules:

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx} \qquad \frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}$$

$y = x^4 + 12x$ $\qquad$ $y = x^4 - 2x^2 + 2$

$y' = 4x^3 + 12$ $\qquad$ $\frac{dy}{dx} = 4x^3 - 4x$

→

product rule:

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$$

Notice that this is not just the product of two derivatives.

This is sometimes memorized as: $d(uv) = du\ v + u\ dv$

$$\frac{d}{dx}\left[\left(x^2+3\right)\left(2x^3+5x\right)\right] = (2x)\left(2x^3+5x\right) + \left(x^2+3\right)\left(6x^2+5\right)$$

$$\frac{d}{dx}\left(2x^5+5x^3+6x^3+15x\right)$$

$$\frac{d}{dx}\left(2x^5+11x^3+15x\right) \qquad 4x^4+10x^2+6x^4+5x^2+18x^2+15$$

$$10x^4+33x^2+15 \qquad 10x^4+33x^2+15$$

→

quotient rule:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$$ or $$d\left(\frac{u}{v}\right) = \frac{du\ v - u\ dv}{v^2}$$

$$\frac{d}{dx}\frac{2x^3+5x}{x^2+3} = \frac{\left(6x^2+5\right)\left(x^2+3\right) - \left[\left(2x^3+5x\right)(2x)\right]}{\left(x^2+3\right)^2}$$

$$= \frac{6x^4+23x^2+15-\left(4x^4-10x^2\right)}{\left(x^2+3\right)^2}$$

$$= \frac{2x^4+13x^2+15}{\left(x^2+3\right)^2}$$

→

Higher Order Derivatives:

$y' = \frac{dy}{dx}$ is the first derivative of y with respect to x.

$y'' = \frac{dy'}{dx} = \frac{d}{dx}\frac{dy}{dx} = \frac{d^2y}{dx^2}$ is the second derivative.

(y double prime)

$y''' = \frac{dy''}{dx}$ is the third derivative.

$y^{(4)} = \frac{d}{dx}y'''$ is the fourth derivative.

We will learn later what these higher order derivatives are used for.

π

Find the horizontal tangents of: $y = x^4 - 2x^2 + 2$

$$\frac{dy}{dx} = 4x^3 - 4x$$

Horizontal tangents occur when slope = zero.

$$4x^3 - 4x = 0$$

$$x^3 - x = 0$$

$$x\left(x^2 - 1\right) = 0$$

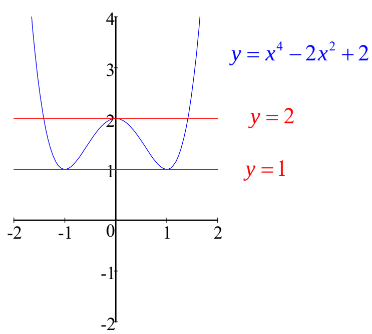
$$x\left(x+1\right)\left(x-1\right) = 0$$

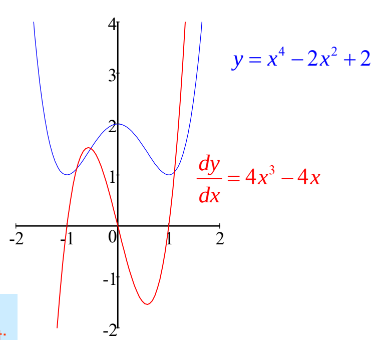
$$x = 0, -1,\ 1$$

Plugging the x values into the original equation, we get:

$$y = 2,\ \ y = 1,\ \ y = 1$$

(The function is even, so we only get two horizontal tangents.)





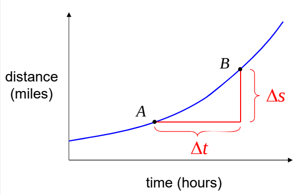
First derivative (slope) is zero at:

$$x = 0, -1,\ 1$$

3.4 Velocity and Other Rates of Change

Objectives:
- Use derivatives to analyze straight line motion and solve other problems involving rates of change.

Consider a graph of displacement (distance traveled) vs. time.



Average velocity can be found by taking:

$$\frac{\text{change in position}}{\text{change in time}} = \frac{\Delta s}{\Delta t}$$

$$V_{\text{ave}} = \frac{\Delta s}{\Delta t} = \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

The speedometer in your car does not measure average velocity, but instantaneous velocity.

(The velocity at one moment in time.)

$$V(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

→

Example: Free Fall Equation

$$s = \frac{1}{2} g\, t^2$$

$$s = \frac{1}{2} \cdot 32\, t^2$$

$$s = 16\, t^2$$

$$V = \frac{ds}{dt} = 32\, t$$

Gravitational Constants:

$$g = 32 \ \frac{\text{ft}}{\text{sec}^2}$$

$$g = 9.8 \ \frac{\text{m}}{\text{sec}^2}$$

$$g = 980 \ \frac{\text{cm}}{\text{sec}^2}$$

Speed is the absolute value of velocity.

→

Acceleration is the derivative of velocity.

$$a = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$$

example: $v = 32t$

$a = 32$

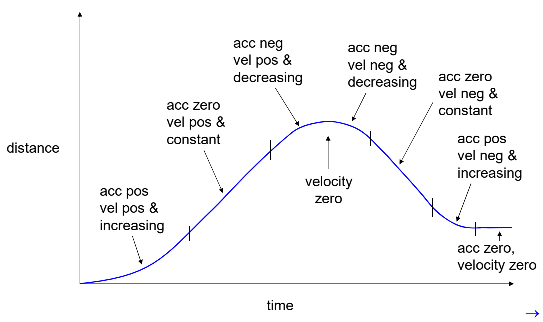
If distance is in: feet

Velocity would be in: $\frac{\text{feet}}{\text{sec}}$

Acceleration would be in: $\frac{\frac{\text{ft}}{\text{sec}}}{\text{sec}} = \frac{\text{ft}}{\text{sec}^2}$

→

It is important to understand the relationship between a position graph, velocity and acceleration:



→

Rates of Change:

Average rate of change = $\frac{f(x+h) - f(x)}{h}$

Instantaneous rate of change = $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

These definitions are true for any function.

(x does not have to represent time.)

→

Example 1:

For a circle:

$A = \pi r^2$

→

from Economics:

Marginal cost is the first derivative of the cost function, and represents an approximation of the cost of producing one more unit.

→

Example 13:

Suppose it costs: $c(x) = x^3 - 6x^2 + 15x$

to produce x stoves.

If you are currently producing 10 stoves, the 11^{th} stove will cost approximately:

→

3.5 A Couple of Jerks

Objectives:
- Use the rules for differentiating the six basic trig functions

A sudden change in acceleration is called a "jerk." When a ride in a car or a bus is jerky, it is not that the accelerations involved are necessarily large but that the changes in acceleration are abrupt. Jerk is what spills your soft drink.

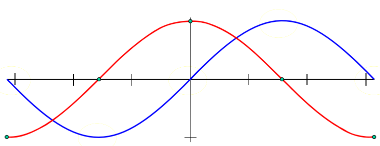
The derivative responsible for jerk is the *third* derivative of position.

Jerk is the derivative of acceleration. If a body's position at time *t* is *s*(*t*), the body's jerk at time *t is*

$$j(t)=\frac{da}{dt}=\frac{d^3s}{dt^2}$$

Consider the function $y=\sin(\theta)$

We could make a graph of the slope:



θ	slope
$-\pi$	-1
$-\frac{\pi}{2}$	0
0	1
$\frac{\pi}{2}$	0
π	-1

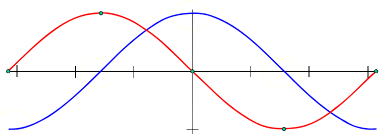
Now we connect the dots!

The resulting curve is a cosine curve.

$$\boxed{\frac{d}{dx}\sin(x)=\cos x}$$

→

We can do the same thing for $y=\cos(\theta)$



θ	slope
$-\pi$	0
$-\frac{\pi}{2}$	1
0	0
$\frac{\pi}{2}$	-1
π	0

The resulting curve is a sine curve that has been reflected about the x-axis.

$$\boxed{\frac{d}{dx}\cos(x)=-\sin x}$$

→

We can find the derivative of tangent *x* by using the quotient rule.

$$\frac{d}{dx}\tan x$$

$$\frac{d}{dx}\frac{\sin x}{\cos x}$$

$$\frac{\cos x\cdot\cos x-\sin x\cdot(-\sin x)}{\cos^2 x}$$

$$\frac{\cos^2 x+\sin^2 x}{\cos^2 x}$$

$$\frac{1}{\cos^2 x}$$

$$\sec^2 x$$

$$\boxed{\frac{d}{dx}\tan(x)=\sec^2 x}$$

→

Derivatives of the remaining trig functions can be determined the same way.

$$\frac{d}{dx}\sin x=\cos x \qquad \frac{d}{dx}\cot x=-\csc^2 x$$

$$\frac{d}{dx}\cos x=-\sin x \qquad \frac{d}{dx}\sec x=\sec x\cdot\tan x$$

$$\frac{d}{dx}\tan x=\sec^2 x \qquad \frac{d}{dx}\csc x=-\csc x\cdot\cot x$$

π

3.6 Chain Rule

Objectives:
- Differentiate composite functions using the Chain Rule
- Find Slopes of parametrized curves

Consider a simple composite function:

$y = 6x - 10$

$y = 2(3x-5)$

If $u = 3x - 5$

then $y = 2u$

$y = 6x - 10 \quad\rightarrow\quad \dfrac{dy}{dx} = 6$

$y = 2u \quad\rightarrow\quad \dfrac{dy}{du} = 2$

$u = 3x - 5 \quad\rightarrow\quad \dfrac{du}{dx} = 3$

$$6 = 2 \cdot 3$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

one more:

$y = 9x^2 + 6x + 1$

$y = (3x+1)^2$

If $u = 3x + 1$

then $y = u^2$

$y = 9x^2 + 6x + 1 \quad\rightarrow\quad \dfrac{dy}{dx} = 18x + 6$

$y = u^2 \quad\rightarrow\quad \dfrac{dy}{du} = 2u$

$u = 3x + 1 \quad\rightarrow\quad \dfrac{du}{dx} = 3$

$\dfrac{dy}{du} = 2(3x+1)$

$\dfrac{dy}{du} = 6x + 2$

$$18x + 6 = (6x+2) \cdot 3$$

This pattern is called the chain rule.

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

Chain Rule: $\boxed{\dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}}$

If $f \circ g$ is the composite of $y=f(u)$ and $u=g(x)$, then:

$$(f \circ g)' = f'_{\text{at } u=g(x)} \cdot g'_{\text{at } x}$$

example: $f(x)=\sin x \quad g(x)=x^2-4$ Find: $(f \circ g)'$ at $x=2$

$f'(x)=\cos x \quad g'(x)=2x \quad g(2)=4-4=0$

$f'(0)\cdot g'(2)$

$\cos(0)\cdot(2\cdot 2)$

$1\cdot 4 \; = 4$

We could also do it this way:

$f(g(x))=\sin(x^2-4)$

Here is a faster way to find the derivative:

$y=\sin(x^2-4)$

Another example:

$$\frac{d}{dx}\cos^2(3x)$$

→

Derivative formulas include the chain rule!

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx} \qquad \frac{d}{dx}\sin u = \cos u\frac{du}{dx}$$

$$\frac{d}{dx}\cos u = -\sin u\frac{du}{dx} \qquad \frac{d}{dx}\tan u = \sec^2 u\frac{du}{dx}$$

etcetera…

The formulas on the memorization sheet are written with u' instead of $\frac{du}{dx}$. Don't forget to include the u' term!

→

The most common mistake on the chapter 3 test is to forget to use the chain rule.

Every derivative problem could be thought of as a chain-rule problem:

$$\frac{d}{dx}x^2 = \underbrace{2x}_{\text{derivative of outside function}}\underbrace{\frac{d}{dx}x}_{\text{derivative of inside function}} = 2x\cdot 1 = 2x$$

The derivative of x is one.

→

The chain rule enables us to find the slope of parametrically defined curves:

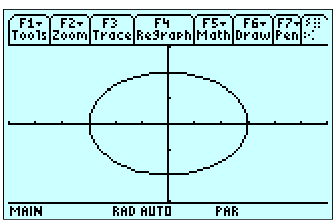
$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx}$$

The slope of a parametrized curve is given by:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

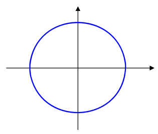
Example: $x = 3\cos t$ $y = 2\sin t$



3.7 Implicit Differentiation

Objectives: •Find derivatives using implicit differentiation

$x^2 + y^2 = 1$



This is not a function, but it would still be nice to be able to find the slope.

$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}1$ ← Do the same thing to both sides.

$2y = x^2 + \sin y$ ← This can't be solved for y.

This technique is called <u>implicit differentiation.</u>

1. Differentiate both sides w.r.t. x.
2. Solve for $\frac{dy}{dx}$.

Find the equations of the lines tangent and normal to the curve $x^2 - xy + y^2 = 7$ at $(-1, 2)$.

We need the slope. Since we can't solve for y, we use implicit differentiation to solve for $\frac{dy}{dx}$.

Find the equations of the lines tangent and normal to the curve $x^2 - xy + y^2 = 7$ at $(-1, 2)$.

$m = \frac{4}{5}$ tangent: normal:

→

Higher Order Derivatives

Find $\frac{d^2y}{dx^2}$ if $2x^3 - 3y^2 = 7$.

→

3.8 Derivatives of Inverse Trig Functions

Objectives: •Calculate derivatives of functions involving the inverse trig functions

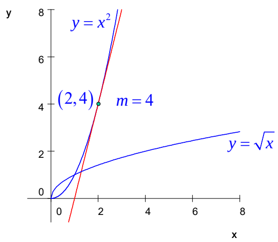
$f(x) = x^2 \quad x \ge 0$

$\dfrac{df}{dx} = 2x$

At $x = 2$:

$f(2) = 2^2 = 4$

$\dfrac{df}{dx}(2) = 2 \cdot 2 = 4$



We can find the inverse function as follows:

$y = x^2$

$x = y^2$

$\sqrt{x} = y$

$y = \sqrt{x}$

$f^{-1}(x) = \sqrt{x}$

To find the derivative of the inverse function:

$f^{-1}(x) = x^{\frac{1}{2}}$

$\dfrac{df^{-1}}{dx} = \dfrac{1}{2} x^{-\frac{1}{2}}$

$\dfrac{df^{-1}}{dx} = \dfrac{1}{2\sqrt{x}}$

→

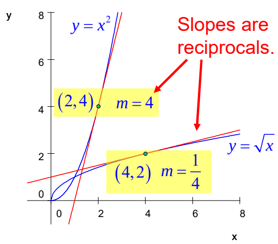
$f(x) = x^2 \quad x \ge 0$

$\dfrac{df}{dx} = 2x$

At $x = 2$:

$f(2) = 2^2 = 4$

$\dfrac{df}{dx}(2) = 2 \cdot 2 = 4$



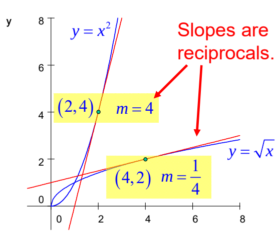
$f^{-1}(x) = \sqrt{x}$

$\dfrac{df^{-1}}{dx} = \dfrac{1}{2\sqrt{x}}$

At $x = 4$:

$f^{-1}(4) = \sqrt{4} = 2$

$\dfrac{df^{-1}}{dx}(4) = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{2 \cdot 2} = \dfrac{1}{4}$



Because x and y are reversed to find the reciprocal function, the following pattern always holds:

Derivative Formula for Inverses:

$$\left.\frac{df^{-1}}{dx}\right|_{x=f(a)} = \frac{1}{\left.\dfrac{df}{dx}\right|_{x=a}}$$

The derivative of $f^{-1}(x)$ evaluated at $f(a)$ is equal to the reciprocal of the derivative of $f(x)$ evaluated at a.

→

A typical problem using this formula might look like this:

Given: $f(3)=5 \quad \frac{df}{dx}(3)=6$

Find: $\frac{df^{-1}}{dx}(5)$

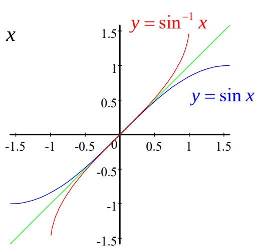
Derivative Formula for Inverses:

$$\left.\frac{df^{-1}}{dx}\right|_{x=f(a)}=\frac{1}{\left.\frac{df}{dx}\right|_{x=a}}$$

→

We can use implicit differentiation to find: $\frac{d}{dx}\sin^{-1}x$

$y=\sin^{-1}x$



We could use the same technique to find $\frac{d}{dx}\tan^{-1}x$ and $\frac{d}{dx}\sec^{-1}x$.

$\csc^{-1}x=\frac{\pi}{2}-\sec^{-1}x$

$$\frac{d}{dx}\sin^{-1}u=\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}\tan^{-1}u=\frac{1}{1+u^2}\frac{du}{dx}$$

$$\frac{d}{dx}\sec^{-1}u=\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$

$$\frac{d}{dx}\cos^{-1}u=-\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}\cot^{-1}u=-\frac{1}{1+u^2}\frac{du}{dx}$$

$$\frac{d}{dx}\csc^{-1}u=-\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$

→

Your calculator contains all six inverse trig functions.
However it is occasionally still useful to know the following:

$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$$

$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$$

$$\csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$$

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Using the Formulas

$$\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}\left(\sin^{-1} x^2\right)$$

Using the Formulas

$$\frac{d}{dx}\sec^{-1} u = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$

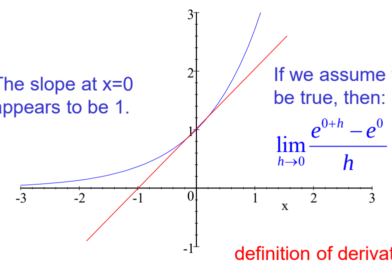
$$\frac{d}{dx}\sec^{-1}\left(5x^4\right)$$

3.9 Derivatives of Exponential and Logarithmic Functions

Objectives: •Calculate derivatives of exponential and logarithmic functions

Look at the graph of $y = e^x$

The slope at x=0 appears to be 1.



If we assume this to be true, then:

$$\lim_{h \to 0} \frac{e^{0+h} - e^0}{h} = 1$$

definition of derivative

→

Now we attempt to find a general formula for the derivative of $y = e^x$ using the definition.

$$\frac{d}{dx}\left(e^x\right) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

→

e^x is its own derivative!

If we incorporate the chain rule:

$$\boxed{\frac{d}{dx}e^u = e^u \frac{du}{dx}}$$

We can now use this formula to find the derivative of a^x

→

$$\frac{d}{dx}\left(a^x\right)$$

$$\frac{d}{dx}\left(e^{\ln a^x}\right)$$ (e^x and $\ln x$ are inverse functions.)

$$\frac{d}{dx}\left(e^{x\ln a}\right)$$

$$e^{x\ln a} \cdot \frac{d}{dx}\left(x \ln a\right)$$ (chain rule)

→

$$\frac{d}{dx}\left(a^x\right) \longrightarrow e^{x\ln a} \cdot \ln a$$

$$a^x \cdot \ln a$$

$$\frac{d}{dx}\left(e^{\ln a^x}\right)$$

$$\frac{d}{dx}\left(e^{x\ln a}\right)$$

$$e^{x\ln a} \cdot \frac{d}{dx}\left(x \ln a\right)$$

Incorporating the chain rule:

$$\boxed{\frac{d}{dx}\left(a^u\right) = a^u \ln a \frac{du}{dx}}$$

→

So far today we have:

$$\frac{d}{dx}e^u = e^u \frac{du}{dx}$$

$$\frac{d}{dx}\left(a^u\right) = a^u \ln a \frac{du}{dx}$$

Now it is relatively easy to find the derivative of $\ln x$.

→

$$y = \ln x \qquad \frac{dy}{dx} = \frac{1}{e^y}$$

$$e^y = x$$

$$\frac{d}{dx}\left(e^y\right) = \frac{d}{dx}(x) \qquad \frac{d}{dx}\ln x = \frac{1}{x}$$

$$e^y \frac{dy}{dx} = 1$$

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$$

→

$$\frac{d}{dx}\log x = \frac{d}{dx}\frac{\ln x}{\ln 10} = \frac{1}{\ln 10}\frac{d}{dx}\ln x = \frac{1}{\ln 10}\cdot\frac{1}{x}$$

The formula for the derivative of a log of any base other than e is:

$$\frac{d}{dx}\log_a u = \frac{1}{u \ln a}\frac{du}{dx}$$

→

$$\frac{d}{dx}e^u = e^u \frac{du}{dx}$$

$$\frac{d}{dx}\left(a^u\right) = a^u \ln a \frac{du}{dx}$$

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$$

$$\frac{d}{dx}\log_a u = \frac{1}{u \ln a}\frac{du}{dx}$$

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4.1 Extreme Value Functions

Objectives: •Determine the local and global extreme values of a function

The textbook gives the following example at the start of chapter 4:

The mileage of a certain car can be approximated by:

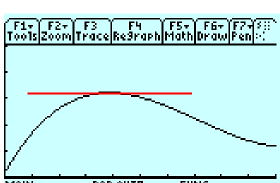
$$m(v) = 0.00015v^3 - 0.032v^2 + 1.8v + 1.7$$

At what speed should you drive the car to obtain the best gas mileage?

Of course, this problem isn't entirely realistic, since it is unlikely that you would have an equation like this for your car.

$$m(v) = 0.00015v^3 - 0.032v^2 + 1.8v + 1.7$$

Notice that at the top of the curve, the horizontal tangent has a slope of zero.



Traditionally, this fact has been used both as an aid to graphing by hand and as a method to find maximum (and minimum) values of functions.

Even though the graphing calculator and the computer have eliminated the need to routinely use calculus to graph by hand and to find maximum and minimum values of functions, we still study the methods to increase our understanding of functions and the mathematics involved.

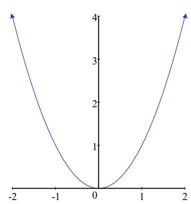
Absolute extreme values are either maximum or minimum points on a curve.

They are sometimes called global extremes.

They are also sometimes called absolute extrema. (*Extrema* is the plural of the Latin *extremum*.)

→

Extreme values can be in the interior or the end points of a function.



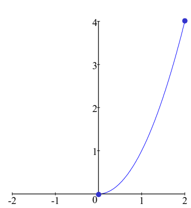
No Absolute Maximum

$$y = x^2$$

$$D = (-\infty, \infty)$$

Absolute Minimum

→



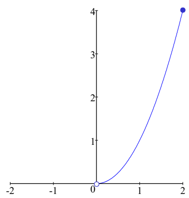
Absolute Maximum

$$y = x^2$$

$$D = [0, 2]$$

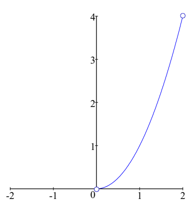
Absolute Minimum

→



$y = x^2$

$D = (0, 2]$

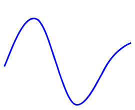


$y = x^2$

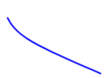
$D = (0, 2)$

Extreme Value Theorem:

If f is continuous over a closed interval, then f has a maximum and minimum value over that interval.



Maximum & minimum at interior points



Maximum & minimum at endpoints

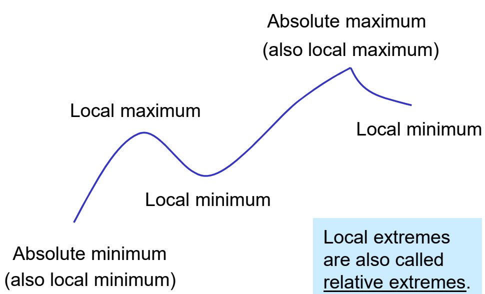
Maximum at interior point, minimum at endpoint

Local Extreme Values:

A local maximum is the maximum value within some open interval.

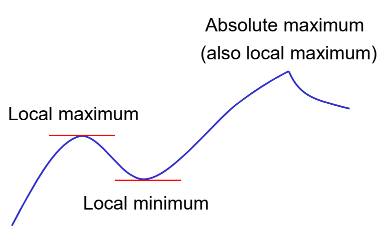
A local minimum is the minimum value within some open interval.

→



Local extremes are also called relative extremes.

→



Notice that local extremes in the interior of the function occur where f' is zero or f' is undefined.

→

Local Extreme Values:

If a function f has a local maximum value or a local minimum value at an interior point c of its domain, and if f' exists at c, then

$$f'(c) = 0$$

→

Critical Point:

A point in the domain of a function f at which $f' = 0$ or f' does not exist is a **critical point** of f.

Note:
Maximum and minimum points in the interior of a function always occur at critical points, but critical points are not always maximum or minimum values.

→

EXAMPLE 3 FINDING ABSOLUTE EXTREMA

Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

$$f(x) = x^{2/3}$$

There are no values of x that will make the first derivative equal to zero.

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$$

The first derivative is undefined at x=0, so (0,0) is a critical point.

$$f'(x) = \frac{2}{3\sqrt[3]{x}}$$

Because the function is defined over a closed interval, we also must check the endpoints.

→

$f(x) = x^{2/3}$ $D = [-2, 3]$

At: $x = 0$ $f(0) = 0$ To determine if this critical point is actually a maximum or minimum, we try points on either side, without passing other critical points.

$f(-1) = 1$ $f(1) = 1$

Since 0<1, this must be at least a local minimum, and possibly a global minimum.

At: $x = -2$ $f(-2) = (-2)^{\frac{2}{3}} \approx 1.5874$

At: $x = 3$ $f(3) = (3)^{\frac{2}{3}} \approx 2.08008$

→

$f(x) = x^{2/3}$ $D = [-2, 3]$

At: $x = 0$ $f(0) = 0$

Absolute minimum: $(0, 0)$

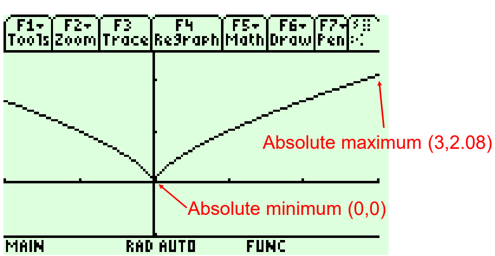
Absolute maximum: $(3, 2.08)$

$f(-1) = 1$ $f(1) = 1$

At: $x = -2$ $f(-2) = (-2)^{\frac{2}{3}} \approx 1.5874$

At: $x = 3$ $f(3) = (3)^{\frac{2}{3}} \approx 2.08008$

→



$f(x) = x^{2/3}$

→

Finding Maximums and Minimums Analytically:

1. Find the derivative of the function, and determine where the derivative is zero or undefined. These are the critical points.

2. Find the value of the function at each critical point.

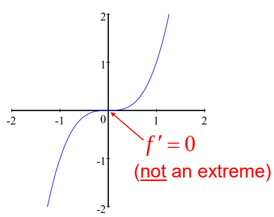
3. Find values or slopes for points between the critical points to determine if the critical points are maximums or minimums.

4. For closed intervals, check the end points as well.

→

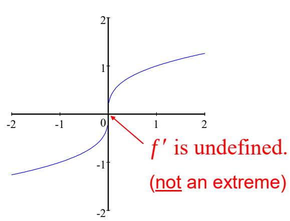
Critical points are not always extremes!

$y = x^3$



→

$y = x^{1/3}$



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4.2 Mean Value Theorem

Objectives: •Apply the Mean Value Theorem to find the intervals on which a function is increasing or decreasing

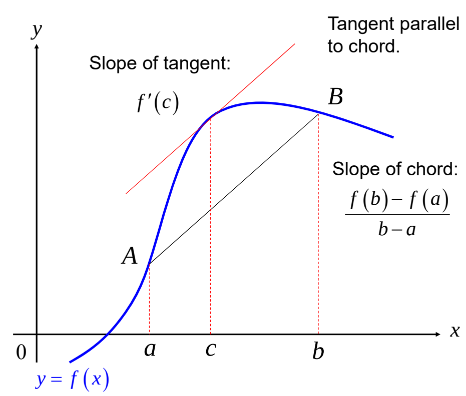
Mean Value Theorem for Derivatives

If $f(x)$ is a differentiable function over $[a,b]$, then at some point between a and b:

$$\frac{f(b)-f(a)}{b-a}=f'(c)$$

The Mean Value Theorem says that **at some point in the closed interval, the actual slope equals the average slope**.

→



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Example 1: Show that the function $f(x)=x^2$ satisfies the hypotheses of the Mean Value Theorem on the interval [0,2]. Then find the value of *c* in the interval that satisfies the equation.

The function is continuous on [0,2] and differentiable on (0,2). Since *f* (0) = 0 and *f* (2) = 4, the Mean Value Theorem guarantees a point *c* in the interval.

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$f'(c)=\frac{f(2)-f(0)}{2-0}$$

$$f'(x)=2x$$

$$f'(c)=2c$$

$$2c=\frac{4-0}{2-0}$$

$$2c=2$$

$$\boxed{c=1}$$

A function is <u>increasing</u> over an interval if the derivative is always positive.

A function is <u>decreasing</u> over an interval if the derivative is always negative.

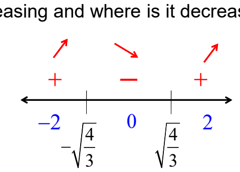
→

Where is $f(x)=x^3-4x$ increasing and where is it decreasing?

$$f'(x)=3x^2-4$$

$$0=3x^2-4$$

$$x=\pm\sqrt{\frac{4}{3}}$$



Increasing:

$$\left(-\infty,-\sqrt{\tfrac{4}{3}}\right),\left(\sqrt{\tfrac{4}{3}},\infty\right)$$

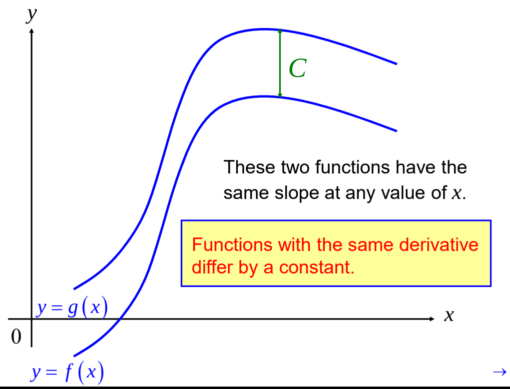
Decreasing:

$$\left(-\sqrt{\tfrac{4}{3}},\sqrt{\tfrac{4}{3}}\right)$$

$$f'(-2)=3(-2)^2-4$$
$$=+8$$

$$f'(0)=3(0)^2-4$$
$$=-4$$

$$f'(2)=3(2)^2-4$$
$$=+8$$



Example 6:

Find the function $f(x)$ whose derivative is $\sin(x)$ and whose graph passes through $(0,2)$.

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

so: $$\frac{d}{dx} - \cos(x) = \sin(x)$$

$$\therefore\ f(x) = -\cos(x) + C$$

$$2 = -\cos(0) + C$$

$$2 = -1 + C$$

$$3 = C$$

Notice that we had to have initial values to determine the value of C.

$$f(x) = -\cos(x) + 3$$

The process of finding the original function from the derivative is so important that it has a name:

Antiderivative

A function $F(x)$ is an **antiderivative** of a function $f(x)$ if $F'(x) = f(x)$ for all x in the domain of f. The process of finding an antiderivative is **antidifferentiation**.

You will hear <u>much</u> more about antiderivatives in the future.

This section is just an introduction.

Example 7b: Find the velocity and position equations for a downward acceleration of 9.8 m/sec^2 and an initial velocity of 1 m/sec downward.

$$\boxed{a(t)=9.8}$$
$$v(t)=9.8t+C$$
$$1=9.8(0)+C$$
$$1=C$$
$$\boxed{v(t)=9.8t+1}$$

$$s(t)=\frac{9.8}{2}t^2+t+C$$
$$s(t)=4.9t^2+t+C$$

The initial position is zero at time zero.

$$0=4.9(0)^2+0+C$$
$$0=C$$
$$\boxed{s(t)=4.9t^2+t}$$

4.3 Connecting *f'* and *f''* with the Graph of *f*.

Objectives:

- Use the First and Second Derivative Tests to determine the local extreme values of a function
- Determine the concavity of a function and locate the points of inflection by analyzing the 2nd derivative
- Graph *f* using information about *f'*.

First derivative:

y' is positive → Curve is rising.

y' is negative → Curve is falling.

y' is zero → Possible local maximum or minimum.

Second derivative:

y'' is positive → Curve is concave up.

y'' is negative → Curve is concave down.

y'' is zero → Possible inflection point (where concavity changes).

Example:

Graph $y = x^3 - 3x^2 + 4 = (x+1)(x-2)^2$

→

Example:

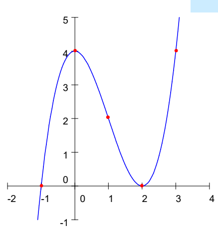
Graph $y = x^3 - 3x^2 + 4 = (x+1)(x-2)^2$

We then look for inflection points by setting the second derivative equal to zero.

→

Make a summary table:

x	y	y'	y''	
-1	0	9	-12	rising, concave down
0	4	0	-6	local max
1	2	-3	0	falling, inflection point
2	0	0	6	local min
3	4	9	12	rising, concave up



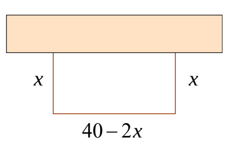
π

4.4 Modeling and Optimization

Objectives: •Solve application problems involving finding minimum or maximum values of functions

A Classic Problem

You have 40 feet of fence to enclose a rectangular garden along the side of a barn. What is the maximum area that you can enclose?



$w = x$

$l = 40 - 2x$

$\boxed{\begin{array}{c} w = 10 \text{ ft} \\ l = 20 \text{ ft} \end{array}}$

$A = x(40 - 2x)$

$A = 40x - 2x^2$

$A' = 40 - 4x$

$0 = 40 - 4x$

$4x = 40$

$x = 10$

$A = 10(40 - 2 \cdot 10)$

$A = 10(20)$

$\boxed{A = 200 \text{ ft}^2}$

→

To find the maximum (or minimum) value of a function:

1. Write it in terms of <u>one</u> variable.
2. Find the first derivative and set it equal to zero.
3. Check the end points if necessary.

→

Example 5: What dimensions for a one liter cylindrical can will use the least amount of material?

We can minimize the material by minimizing the area.

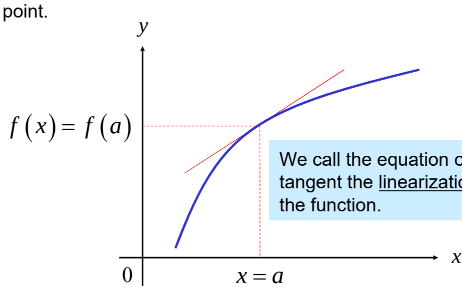
We need another equation that relates r and h:

$$A = \underbrace{2\pi r^2}_{\text{area of ends}} + \underbrace{2\pi rh}_{\text{lateral area}}$$

4.5 Linearization and Newton's Method

Objectives:
- Find linearizations
- Estimate the change in a function using differentials

For any function $f(x)$, the tangent is a close approximation of the function for some small distance from the tangent point.



We call the equation of the tangent the linearization of the function.

Start with the point/slope equation:

$y - y_1 = m(x - x_1) \qquad x_1 = a \qquad y_1 = f(a) \qquad m = f'(a)$

$y - f(a) = f'(a)(x - a)$

$y = f(a) + f'(a)(x - a)$

$\boxed{L(x) = f(a) + f'(a)(x - a)}$ linearization of f at a

$f(x) \approx L(x)$ is the standard linear approximation of f at a.

The linearization is the equation of the tangent line, and you can use the old formulas if you like.

→

Example: Find the linearization of $f(x) = \sqrt{1+x}$ at $x = 0$, and use it to approximate $\sqrt{1.02}$ without a calculator.

$f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$

$= \dfrac{1}{2\sqrt{1+x}}$

$f'(0) = \dfrac{1}{2}$

$L(x) = f(a) + f'(a)(x - a)$

$L(x) = 1 + \frac{1}{2}(x - 0)$

$= 1 + \dfrac{x}{2}$

$L(.02) = 1 + \dfrac{.02}{2}$

$= 1.01$

Example: Find the linearization of $f(x) = \cos x$ at $x = \frac{\pi}{2}$ and use it to approximate $\cos 1.75$

$f(\frac{\pi}{2}) = 0$

$f'(x) = -\sin x$

$f'(\frac{\pi}{2}) = -\sin(\frac{\pi}{2})$

$= -1$

$L(x) = f(a) + f'(a)(x - a)$

$L(x) = 0 - 1(x - \frac{\pi}{2})$

$= -x + \dfrac{\pi}{2}$

$L(1.75) = -1.75 + \frac{\pi}{2}$

$\approx -.1792$

Use linearization's to approximate $\sqrt{123}$.

$f(x)=\sqrt{x}$

Let $x=121$

$f(121)=11$

$f'(x)=\dfrac{1}{2\sqrt{x}}$

$f'(121)=\dfrac{1}{22}$

$L(x)=11+\dfrac{1}{22}(x-121)$

$L(123)=11+\dfrac{1}{22}(123-121)$

$L(123)=11+\dfrac{1}{11}$

$=11.09$

Important linearizations for x near zero:

$f(x)$	$L(x)$
$(1+x)^k$	$1+kx$
$\sin x$	x
$\cos x$	1
$\tan x$	x

→

Differentials:

When we first started to talk about derivatives, we said that $\frac{\Delta y}{\Delta x}$ becomes $\frac{dy}{dx}$ when the change in x and change in y become very small.

↓

dy can be considered a very small change in y.

dx can be considered a very small change in x.

→

Let $y = f(x)$ be a differentiable function.
The differential dx is an independent variable.
The differential dy is: $dy = f'(x)\,dx$

→

Example: Consider a circle of radius 10. If the radius increases by 0.1, approximately how much will the area change?

$$A = \pi r^2 \qquad \frac{dA}{\cancel{dx}} = 2\pi r\,\frac{dr}{\cancel{dx}}$$

$$dA = 2\pi r\ dr$$

very small change in r

very small change in A

$$dA = 2\cdot\pi\cdot 10\cdot(0.1)$$

$dA = 2\pi$ (approximate change in area)

→

$dA = 2\pi$ (approximate change in area)

Compare to actual change:

New area: $\pi(10.1)^2 = 102.01\pi$

Old area: $\pi(10)^2 = \underline{100.00\pi}$

2.01π

$$\frac{\text{Error}}{\text{Actual Answer}} = \frac{.01\pi}{2.01\pi} \approx .0049751 \approx 0.5\%$$

$$\frac{\text{Error}}{\text{Original Area}} = \frac{.01\pi}{100\pi} \approx .0001 \approx 0.01\%$$

→

4.6 Related Rates

Objectives: •Solve related rate problems

First, a review problem:

Consider a sphere of radius 10cm.

If the radius changes 0.1cm (a very small amount) how much does the volume change?

$$V = \frac{4}{3}\pi r^3$$

$$dV = 4\pi r^2 dr$$

$$dV = 4\pi (10\text{cm})^2 \cdot 0.1\text{cm}$$

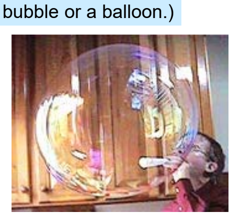
$$dV = 40\pi \text{cm}^3$$

The volume would change by approximately $40\pi \text{cm}^3$.

→

Now, suppose that the radius is changing at an instantaneous rate of 0.1 cm/sec.

(Possible if the sphere is a soap bubble or a balloon.)



$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi (10\text{cm})^2 \cdot \left(0.1\frac{\text{cm}}{\text{sec}}\right)$$

$$\frac{dV}{dt} = 40\pi \frac{\text{cm}^3}{\text{sec}}$$

The sphere is growing at a rate of $40\pi \text{ cm}^3/\text{sec}$.

Note: This is an exact answer, not an approximation like we got with the differential problems.

→

Water is draining from a cylindrical tank at 3 liters/second. How fast is the surface dropping?

$$\frac{dV}{dt} = -3\,\frac{\text{L}}{\text{sec}} = -3000\,\frac{\text{cm}^3}{\text{sec}}$$

Find $\frac{dh}{dt}$

$$V = \pi r^2 h$$

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$$

$$-3000\,\frac{\text{cm}^3}{\text{sec}} = \pi r^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = -\frac{3000\,\frac{\text{cm}^3}{\text{sec}}}{\pi r^2}$$

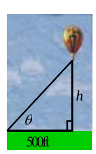
→

Steps for Related Rates Problems:

1. Draw a picture (sketch).
2. Write down known information.
3. Write down what you are looking for.
4. Write an equation to relate the variables.
5. Differentiate both sides with respect to *t*.
6. Evaluate.

→

Hot Air Balloon Problem:



Given: $\theta = \frac{\pi}{4}$ $\quad \frac{d\theta}{dt} = 0.14\,\frac{\text{rad}}{\text{min}}$

How fast is the balloon rising?

Find $\frac{dh}{dt}$

$$\sec\frac{\pi}{4} = \sqrt{2}$$

$$\tan\theta = \frac{h}{500}$$

$$\sec^2\theta\,\frac{d\theta}{dt} = \frac{1}{500}\frac{dh}{dt}$$

$$\left(\sqrt{2}\right)^2(0.14)\cdot 500 = \frac{dh}{dt}$$

$$\left(\sec\frac{\pi}{4}\right)^2(0.14) = \frac{1}{500}\frac{dh}{dt}$$

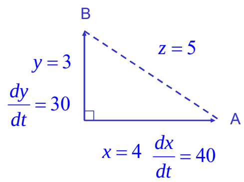
$$\boxed{140\,\frac{\text{ft}}{\text{min}} = \frac{dh}{dt}}$$

→

Truck Problem:

Truck A travels east at 40 mi/hr.
Truck B travels north at 30 mi/hr.

How fast is the distance between the trucks changing 6 minutes later?



$$x^2+y^2=z^2$$

$$\cancel{2}x\frac{dx}{dt}+\cancel{2}y\frac{dy}{dt}=\cancel{2}z\frac{dz}{dt}$$

$$4\cdot 40+3\cdot 30=5\frac{dz}{dt}$$

$$250=5\frac{dz}{dt}$$

$$50=\frac{dz}{dt}$$

$$\boxed{50\frac{\text{miles}}{\text{hour}}}$$

π

5.1 Estimating with Finite Sums

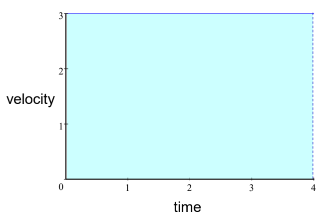
Objectives:
- Approximate the area under the graph of a nonnegative continuous function by using rectangle approximation methods
- Interpret the area under a graph as a net accumulation of a rate of change

---

Consider an object moving at a constant rate of 3 ft/sec.

Since rate · time = distance: $3t = d$

If we draw a graph of the velocity, the distance that the object travels is equal to the area under the line.



After 4 seconds, the object has gone 12 feet.

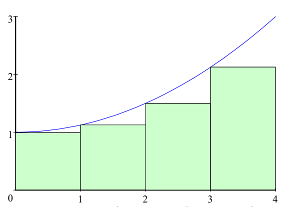
$$3\frac{\text{ft}}{\text{sec}} \cdot 4\text{ sec} = 12\text{ ft}$$

→

If the velocity is not constant, we might guess that the distance traveled is still equal to the area under the curve.

(The units work out.)

Example: $V = \frac{1}{8}t^2 + 1$



$1 \quad 1\frac{1}{8} \quad 1\frac{1}{2} \quad 2\frac{1}{8}$

t	v
0	1
1	$1\frac{1}{8}$
2	$1\frac{1}{2}$
3	$2\frac{1}{8}$

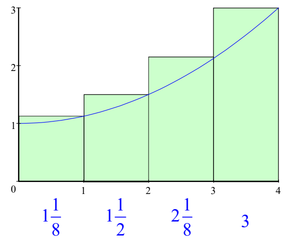
We could estimate the area under the curve by drawing rectangles touching at their left corners.

This is called the Left-hand Rectangular Approximation Method (LRAM).

Approximate area: $1 + 1\frac{1}{8} + 1\frac{1}{2} + 2\frac{1}{8} = 5\frac{3}{4} = 5.75$

→

$V = \frac{1}{8}t^2 + 1$

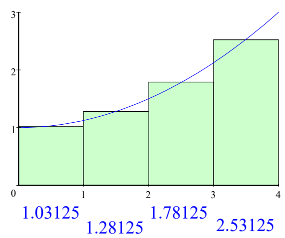


We could also use a Right-hand Rectangular Approximation Method (RRAM).

Approximate area: $1\frac{1}{8} + 1\frac{1}{2} + 2\frac{1}{8} + 3 = 7\frac{3}{4} = 7.75$

$V = \frac{1}{8}t^2 + 1$

t	v
0.5	1.03125
1.5	1.28125
2.5	1.78125
3.5	2.53125



Another approach would be to use rectangles that touch at the midpoint. This is the Midpoint Rectangular Approximation Method (MRAM).

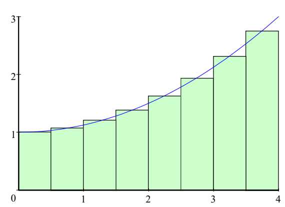
Approximate area: 6.625

In this example there are four subintervals.
As the number of subintervals increases, so does the accuracy.

With 8 subintervals:

$V = \frac{1}{8}t^2 + 1$

t	v
0.25	1.00781
0.75	1.07031
1.25	1.19531
1.75	1.38281
2.25	1.63281
2.75	1.94531
3.25	2.32031
3.75	2.75781



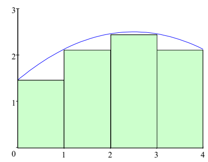
Approximate area: 6.65624

$13.31248 \times 0.5 = 6.65624$

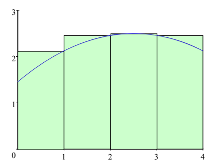
width of subinterval

The exact answer for this problem is $6.\overline{6}$.

Inscribed rectangles are all below the curve:



Circumscribed rectangles are all above the curve:



→

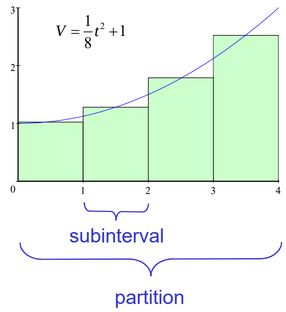
We will be learning how to find the exact area under a curve if we have the equation for the curve. Rectangular approximation methods are still useful for finding the area under a curve if we do not have the equation.

The TI-89 calculator can do these rectangular approximation problems. This is of limited usefulness, since we will learn better methods of finding the area under a curve, but you could use the calculator to check your work.

→

5.2 The Definite Integral

Objectives: •Express the area under a curve as a definite integral and as a limit of Riemann sums.



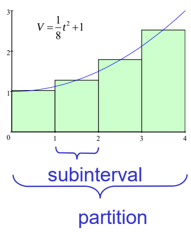
When we find the area under a curve by adding rectangles, the answer is called a **Rieman sum**.

The width of a rectangle is called a **subinterval**.

The entire interval is called the **partition**.

Subintervals do not all have to be the same size.

→



If the partition is denoted by P, then the length of the longest subinterval is called the **norm** of P and is denoted by $\|P\|$.

As $\|P\|$ gets smaller, the approximation for the area gets better.

$$\text{Area} = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f\left(c_k\right) \Delta x_k$$ if P is a partition of the interval $\left[a,b\right]$

→

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f\left(c_k\right) \Delta x_k$$ is called the **definite integral** of f over $\left[a,b\right]$.

If we use subintervals of equal length, then the length of a subinterval is: $\Delta x = \dfrac{b-a}{n}$

The definite integral is then given by:

$$\lim_{n \to \infty} \sum_{k=1}^{n} f\left(c_k\right) \Delta x$$

→

$$\boxed{\lim_{n\to\infty}\sum_{k=1}^{n} f\left(c_k\right)\Delta x}$$

Leibnitz introduced a simpler notation for the definite integral:

$$\lim_{n\to\infty}\sum_{k=1}^{n} f\left(c_k\right)\Delta x = \int_a^b f\left(x\right)dx$$

Note that the very small change in x becomes dx.

→

---

upper limit of integration

Integration Symbol

$$\int_a^b \underbrace{f\left(x\right)}_{\text{integrand}} dx$$

lower limit of integration

variable of integration (dummy variable)

It is called a dummy variable because the answer does not depend on the variable chosen.

→

Definition Area Under a Curve (as a Definite Integral)

If $y = f(x)$ is a nonnegative and integrable over a closed interval $[a, b]$, then the <u>area under the curve $y = f(x)$ from a to b</u> is the integral of f from a to b,

$$\int_a^b f\left(x\right)dx$$

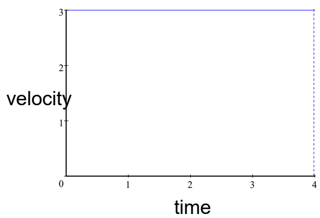
We have the notation for integration, but we still need to learn how to evaluate the integral.

→

In section 5.1, we considered an object moving at a constant rate of 3 ft/sec.

Since rate · time = distance: $3t = d$

If we draw a graph of the velocity, the distance that the object travels is equal to the area under the line.



After 4 seconds, the object has gone 12 feet.

$$3\,\frac{\text{ft}}{\text{sec}}\cdot 4\ \text{sec} = 12\ \text{ft}$$

This is also the same as saying $\int_0^4 3dx = 12$

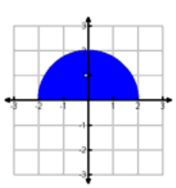
The Integral of a Constant

If $f(x) = c$, where c is a constant, on the interval $[a, b]$, then

$$\int_a^b c\ dx = c(b-a)$$

Evaluate the integral $\int_{-2}^{2} \sqrt{4-x^2}\,dx = 2\pi$

To evaluate this integral, we can graph the function $\sqrt{4-x^2}$ and then find the area under the curve



The curve is a semi-circle with a radius of 2.

$$A = \frac{1}{2}\pi r^2$$

$$A = \frac{1}{2}\pi (2)^2$$

$$A = 2\pi$$

Other Important Information

When $f(x) \le 0$, the function is below the x –axis, therefore the area is negative.

The area of a trapezoid is $A = \frac{1}{2}(b_1 + b_2)h$

5.3 Definite Integrals and Antiderivatives

Objectives: •Apply rules for definite integrals.

•Find the average value of a function over a closed interval.

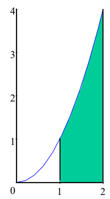
$$\text{Area} = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k$$

$$= \int_a^b f(x)\,dx$$

$$= F(b) - F(a)$$

$F(x)$ is the antiderivative of $f(x)$

Example: $y = x^2$



Find the area under the curve from $x=1$ to $x=2$.

$$\int_1^2 x^2 dx$$

$$\left.\frac{1}{3}x^3\right|_1^2$$

$$\frac{1}{3}\cdot 2^3 - \frac{1}{3}\cdot 1$$

$$\frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

Area under the curve from $x=1$ to $x=2$.

→

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ Reversing the limits changes the sign.

2. $\int_a^a f(x)\,dx = 0$ If the upper and lower limits are equal, then the integral is zero.

3. $\int_a^b k\cdot f(x)\,dx = k\int_a^b f(x)\,dx$ Constant multiples can be moved outside.

4. $\int_a^b \left[f(x)+g(x)\right]dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

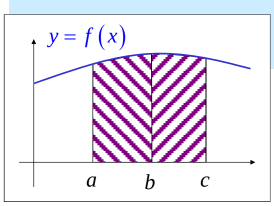
Integrals can be added and subtracted.

→

4. $\int_a^b \left[f(x)+g(x)\right]dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

Integrals can be added and subtracted.

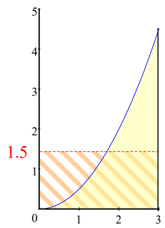
5. $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$



<u>Intervals</u> can be added (or subtracted.)

→

The <u>average value</u> of a function is the value that would give the same area if the function was a constant:



$y = \frac{1}{2}x^2$

$$A = \int_0^3 \frac{1}{2}x^2 dx$$

$$= \frac{1}{6}x^3 \Big|_0^3 \quad = \frac{27}{6} \quad = \frac{9}{2} \quad = 4.5$$

$$\text{Average Value} = \frac{4.5}{3} = 1.5$$

$$\text{Average Value} = \frac{\text{Area}}{\text{Width}} = \frac{1}{b-a}\int_a^b f(x)\ dx$$

The mean value theorem for definite integrals says that for a continuous function, at some point on the interval the actual value will equal the average value.

Mean Value Theorem (for definite integrals)

If f is continuous on $[a,b]$ then at some point c in $[a,b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\ dx$$

5.4 Fundamental Theorem of Calculus

Objectives: •Apply the Fundamental Theorem of Calculus

•Understand the relationship between the derivative and the definite integral as expressed in both parts of the FTC

The Fundamental Theorem of Calculus, Part 1

If f is continuous on $[a,b]$, then the function

$$F(x)=\int_a^x f(t)\,dt$$

has a derivative at every point in $[a,b]$, and

$$\frac{dF}{dx}=\frac{d}{dx}\int_a^x f(t)\,dt=f(x)$$

→

First Fundamental Theorem:

$$\frac{d}{dx}\int_a^x f(t)\,dt=f(x)$$

1. Derivative of an integral.
2. Derivative matches upper limit of integration.
3. Lower limit of integration is a constant.

→

$$\frac{d}{dx}\int_{-\pi}^x \cos t\;dt=\cos x$$

$$\frac{d}{dx}\left(\sin t\big|_{-\pi}^{x}\right)$$

$$\frac{d}{dx}\left(\sin x-\cancelto{0}{\sin(-\pi)}\right)$$

$$\frac{d}{dx}\sin x$$

$$\cos x$$

First Fundamental Theorem:

1. Derivative of an integral.
2. Derivative matches upper limit of integration.
3. Lower limit of integration is a constant.

→

$$\frac{d}{dx}\int_0^x \frac{1}{1+t^2}\,dt \quad = \frac{1}{1+x^2}$$

1. Derivative of an integral.
2. Derivative matches upper limit of integration.
3. Lower limit of integration is a constant.

→

$$\frac{d}{dx}\int_0^{x^2} \cos t\ dt$$

The upper limit of integration does <u>not</u> match the derivative, but we could use the **chain rule**.

$$\cos\left(x^2\right)\cdot\frac{d}{dx}x^2$$

$$\cos\left(x^2\right)\cdot 2x$$

$$2x\cos\left(x^2\right)$$

→

$$\frac{d}{dx}\int_x^5 3t\sin t\ dt$$

The lower limit of integration is not a constant, but the upper limit is.

We can **change the sign** of the integral and **reverse the limits**.

$$-\frac{d}{dx}\int_5^x 3t\sin t\ dt$$

$$-3x\sin x$$

→

$$\frac{d}{dx}\int_{2x}^{x^2}\frac{1}{2+e^t}\,dt$$

Neither limit of integration is a constant.

We **split the integral into two parts**.

$$\frac{d}{dx}\left(\int_0^{x^2}\frac{1}{2+e^t}\,dt+\int_{2x}^{0}\frac{1}{2+e^t}\,dt\right)$$

$$\frac{d}{dx}\left(\int_0^{x^2}\frac{1}{2+e^t}\,dt-\int_{0}^{2x}\frac{1}{2+e^t}\,dt\right)$$

$$\frac{1}{2+e^{x^2}}\cdot 2x-\frac{1}{2+e^{2x}}\cdot 2 \quad =\frac{2x}{2+e^{x^2}}-\frac{2}{2+e^{2x}}$$

→

The Fundamental Theorem of Calculus, Part 2

If f is continuous at every point of $[a,b]$, and if F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx=F(b)-F(a)$$

(Also called the **Integral Evaluation Theorem**)

We already know this!

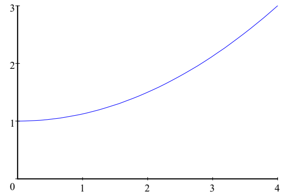
To evaluate an integral, take the anti-derivatives and subtract.

π

5.5 Trapezoidal Rule

Objectives: •Approximate the definite integral by using the Trapezoid Rule and by using Simpson's Rule, and estimate the error in using the Trap and Simpson's Rule.

$$y = \frac{1}{8}x^2 + 1 \qquad 0 \le x \le 4$$



Actual area under curve:

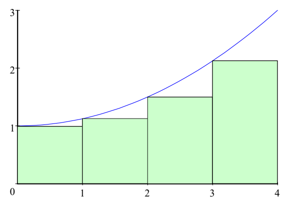
$$A = \int_0^4 \frac{1}{8}x^2 + 1\, dx$$

$$A = \frac{1}{24}x^3 + x \Big|_0^4$$

$$A = \frac{20}{3} = 6.\overline{6}$$

→

$$y = \frac{1}{8}x^2 + 1 \qquad 0 \le x \le 4$$



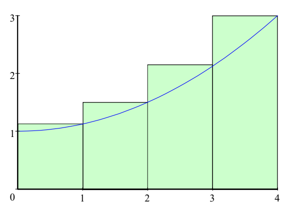
Left-hand rectangular approximation:

Approximate area: $1 + 1\frac{1}{8} + 1\frac{1}{2} + 2\frac{1}{8} = 5\frac{3}{4} = 5.75$

(too low)

→

$$y = \frac{1}{8}x^2 + 1 \qquad 0 \le x \le 4$$



Right-hand rectangular approximation:

Approximate area: $1\frac{1}{8} + 1\frac{1}{2} + 2\frac{1}{8} + 3 = 7\frac{3}{4} = 7.75$

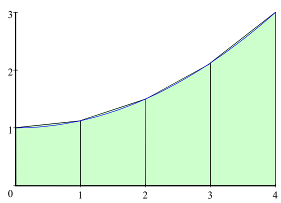
(too high)

→

Averaging the two:

$$\frac{7.75+5.75}{2}=6.75$$ 1.25% error (too high)

→



$$T=\frac{1}{2}\left(1+\frac{9}{8}\right)+\frac{1}{2}\left(\frac{9}{8}+\frac{3}{2}\right)+\frac{1}{2}\left(\frac{3}{2}+\frac{17}{8}\right)+\frac{1}{2}\left(\frac{17}{8}+3\right)$$

$$T=\frac{1}{2}\left(1+\frac{9}{8}+\frac{9}{8}+\frac{3}{2}+\frac{3}{2}+\frac{17}{8}+\frac{17}{8}+3\right)$$

$$T=\frac{1}{2}\left(\frac{27}{2}\right)\quad=\frac{27}{4}\quad=6.75$$ (still too high)

→

Trapezoidal Rule:

$$T=\frac{h}{2}\left(y_0+2y_1+2y_2+\ldots+2y_{n-1}+y_n\right)$$

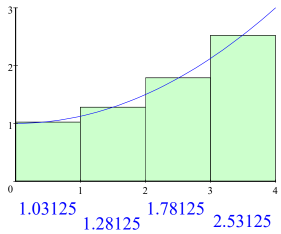
(h = width of subinterval)

This gives us a better approximation than either left or right rectangles.

→

$$y = \frac{1}{8}x^2 + 1 \qquad 0 \le x \le 4$$

Compare this with the Midpoint Rule:



Approximate area: 6.625 0.625% error (too low)

The midpoint rule gives a closer approximation than the trapezoidal rule, but in the opposite direction.

→

Trapezoidal Rule: 6.750 1.25% error (too high)

Midpoint Rule: 6.625 0.625% error (too low)

Notice that the trapezoidal rule gives us an answer that has <u>twice</u> as much error as the midpoint rule, but in the opposite direction.

Ahhh!

If we use a weighted average:

$$\frac{2(6.625)+6.750}{3} = 6.\bar{6}$$

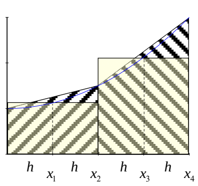
This is the exact answer!

Oooh!

Wow!

→

This weighted approximation gives us a closer approximation than the midpoint or trapezoidal rules.



Midpoint:

$$M = 2h \cdot y_1 + 2h \cdot y_3 = 2h(y_1 + y_3)$$

Trapezoidal:

$$T = \frac{1}{2}(y_0 + y_2)2h + \frac{1}{2}(y_2 + y_4)2h$$

$$T = h(y_0 + y_2) + h(y_2 + y_4)$$

$$T = h(y_0 + 2y_2 + y_4)$$

$$\frac{2M + T}{3}$$

$$= \frac{1}{3}\Big[\underbrace{4h(y_1 + y_3)}_{\text{twice midpoint}} + \underbrace{h(y_0 + 2y_2 + y_4)}_{\text{trapezoidal}}\Big] = \frac{h}{3}\left[4y_1 + 4y_3 + y_0 + 2y_2 + y_4\right]$$

$$= \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4)$$

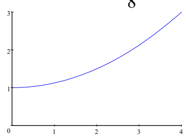
→

Simpson's Rule:

$$S=\frac{h}{3}\left(y_0+4y_1+2y_2+4y_3+\ldots+2y_{n-2}+4y_{n-1}+y_n\right)$$

(h = width of subinterval, n must be even)

Example: $y=\frac{1}{8}x^2+1$



$$S=\frac{1}{3}\left(1+4\cdot\frac{9}{8}+2\cdot\frac{3}{2}+4\cdot\frac{17}{8}+3\right)$$

$$=\frac{1}{3}\left(1+\frac{9}{2}+3+\frac{17}{2}+3\right)$$

$$=\frac{1}{3}(20)=6.\overline{6}$$

→

6.1 Slope Fields and Euler's Method

Objectives:
- Solve initial value problems
- Construct slope fields using technology and interpret slope fields as visualizations of differential equations.

First, a little review:

Consider: $y = x^2 + 3$ or $y = x^2 - 5$

then: $y' = 2x$ $\quad$ $y' = 2x$

It doesn't matter whether the constant was 3 or -5, since when we take the derivative the constant disappears.

However, when we try to reverse the operation:

Given: $y' = 2x$ find y

$y = x^2 + C$

We don't know what the constant is, so we put "C" in the answer to remind us that there might have been a constant.

→

If we have some more information we can find C.

Given: $y' = 2x$ and $y = 4$ when $x = 1$, find the equation for y.

$$y = x^2 + C$$
$$4 = 1^2 + C$$
$$3 = C$$
$$y = x^2 + 3$$

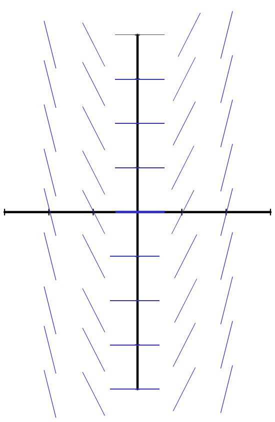
This is called an initial value problem. We need the initial values to find the constant.

An equation containing a derivative is called a differential equation. It becomes an initial value problem when you are given the initial condition and asked to find the original equation.

→

Initial value problems and differential equations can be illustrated with a slope field.

Slope fields are mostly used as a learning tool and are mostly done on a computer or graphing calculator, but a recent AP test asked students to draw a simple one by hand.



$y' = 2x$

x	y	y'
0	0	0
0	1	0
0	2	0
0	3	0
1	0	2
1	1	2
2	0	4
-1	0	-2
-2	0	-4

6.2 Antidifferentiation by Substitution

Objectives: •Compute indefinite integrals by the method of substitution

Example 1:

$$\int (x+2)^5\, dx \qquad \text{Let } u = x+2$$

$$du = dx$$

$$\int u^5 du$$

$$\frac{1}{6}u^6 + C$$

$$\frac{(x+2)^6}{6} + C$$

Don't forget to substitute the value for u back into the problem!

→

Example:
(Exploration 1 in the book)

$$\int \sqrt{1+x^2} \cdot \underline{2x\, dx}$$

One of the clues that we look for is if we can find a function and its derivative in the integrand.

The derivative of $1+x^2$ is $\underline{2x\ dx}$.

$$\int u^{\frac{1}{2}}\, du \qquad \text{Let } u = 1+x^2$$

$$du = 2x\ dx$$

$$\frac{2}{3}u^{\frac{3}{2}} + C$$

$$\frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}} + C$$

Note that this only worked because of the $2x$ in the original.
Many integrals can not be done by substitution.

→

Example 2:

$$\int \sqrt{4x-1}\ dx \qquad \text{Let } u = 4x-1$$

$$\left.\begin{aligned} du &= 4\ dx \\ \frac{1}{4}du &= dx \end{aligned}\right\} \text{Solve for } dx.$$

$$\int u^{\frac{1}{2}} \cdot \frac{1}{4}du$$

$$\frac{2}{3}u^{\frac{3}{2}} \cdot \frac{1}{4} + C$$

$$\frac{1}{6}u^{\frac{3}{2}} + C$$

$$\frac{1}{6}(4x-1)^{\frac{3}{2}} + C$$

→

Example 3:

$$\int \cos(7x+5)\ dx \qquad \text{Let } u = 7x+5$$

$$du = 7\ dx$$

$$\int \cos u \cdot \frac{1}{7} du \qquad \frac{1}{7} du = dx$$

$$\frac{1}{7}\sin u + C$$

$$\frac{1}{7}\sin(7x+5) + C$$

→

Example: (Not in book)

$$\int x^2 \sin(x^3)\ dx \qquad \text{Let } u = x^3$$

$$du = 3x^2\ dx$$

$$\frac{1}{3}\int \sin u\ du \qquad \frac{1}{3} du = \underbrace{x^2\ dx}$$

We solve for $x^2\ dx$ because we can find it in the integrand.

$$-\frac{1}{3}\cos u + C$$

$$-\frac{1}{3}\cos x^3 + C$$

→

Example 7:

$$\int \sin^4 x \cdot \cos x\ dx$$

$$\int (\sin x)^4 \cos x\ dx \qquad \text{Let } u = \sin x$$

$$du = \cos x\ dx$$

$$\int u^4\ du$$

$$\frac{1}{5}u^5 + C$$

$$\frac{1}{5}\sin^5 x + C$$

→

Example:

The technique is a little different for definite integrals.

$$\int_0^{\frac{\pi}{4}} \tan x \ \sec^2 x \ dx$$

Let $u = \tan x$

$du = \sec^2 x \ dx$

$$\int_0^1 u \ du$$

new limit

new limit

$u(0) = \tan 0 = 0$

$u\left(\frac{\pi}{4}\right) = \tan \frac{\pi}{4} = 1$

$$\left.\frac{1}{2}u^2\right|_0^1$$

$$\frac{1}{2}$$

We could have substituted back and used the original limits.

→

Example:

Using the original limits:

$$\int_0^{\frac{\pi}{4}} \tan x \ \sec^2 x \ dx$$

Let $u = \tan x$

$du = \sec^2 x \ dx$

$$\cancel{\int_0^{\frac{\pi}{4}} u \ du}$$

$$\left.\int u \ du \ \right\} \text{Leave the limits out until you substitute back.}$$

$$= \frac{1}{2}u^2$$

$$= \left.\frac{1}{2}(\tan x)^2\right|_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left(\tan \frac{\pi}{4}\right)^2 - \frac{1}{2}(\tan 0)^2$$

$$= \frac{1}{2}\cdot 1^2 - \frac{1}{2}\cdot 0^2 \qquad = \frac{1}{2}$$

This is usually more work than finding new limits

→

Example:

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1} \ dx$$

Let $u = x^3 + 1$ $\qquad u(-1) = 0$

$du = 3x^2 \ dx$ $\qquad u(1) = 2$

$$\int_0^2 u^{\frac{1}{2}} \ du$$

$$\left.\frac{2}{3}u^{\frac{3}{2}}\right|_0^2$$

Don't forget to use the new limits.

$$\frac{2}{3}\cdot 2^{\frac{3}{2}} \quad = \frac{2}{3}\cdot 2\sqrt{2} \quad = \frac{4\sqrt{2}}{3}$$

→

6.3 Integration by Parts

Objectives: •Use integration by parts to evaluate indefinite and definite integrals.
•Use tabular integration or the method of solving for the unknown integral in order to evaluate integrals

6.3 Integration By Parts

Start with the product rule:

$$\frac{d}{\cancel{dx}}(uv) = u\frac{dv}{\cancel{dx}} + v\frac{du}{\cancel{dx}}$$

$$d(uv) = u\ dv + v\ du$$

$$d(uv) - v\ du = u\ dv$$

$$u\ dv = d(uv) - v\ du$$

$$\int u\ dv = \int \left(d(uv) - v\ du\right)$$

$$\int u\ dv = \int \left(d(uv)\right) - \int v\ du$$

$$\boxed{\int u\ dv = uv - \int v\ du}$$

This is the Integration by Parts formula.

→

$$\int u\ dv = uv - \int v\ du$$

u differentiates to zero (usually).

dv is easy to integrate.

The Integration by Parts formula is a "product rule" for integration.

Choose *u* in this order: LIPET

Logs, **I**nverse trig, **P**olynomial, **E**xponential, **T**rig

→

Example 1:

$$\int x \cdot \cos x \, dx$$

$$\int u \, dv = uv - \int v \, du$$

LIPET

→

Example:

$$\int \ln x \, dx$$

$$\int u \, dv = uv - \int v \, du$$

LIPET

→

Example 4:

$$\int x^2 e^x \, dx$$

$$\int u \, dv = uv - \int v \, du$$

LIPET

→

Example 5: LIPET

$$\int e^x \cos x \, dx$$

→

A Shortcut: Tabular Integration

Tabular integration works for integrals of the form:

$$\int \underbrace{f(x)}\,\underbrace{g(x)}\,dx$$

where: Differentiates to zero in several steps. Integrates repeatedly.

→

$$\int x^2 e^x \, dx$$

$f(x)$ & deriv.	$g(x)$ & integrals

→

$$\int x^3 \sin x \, dx$$

π

6.4 Exponential Growth and Decay

Objectives: •Solve problems involving exponential growth and decay in variety of applications

The number of rabbits in a population increases at a rate that is proportional to the number of rabbits present (at least for awhile.)

So does any population of living creatures. Other things that increase or decrease at a rate proportional to the amount present include radioactive material and money in an interest-bearing account.

If the rate of change is proportional to the amount present, the change can be modeled by:

$$\frac{dy}{dt} = ky$$

→

$$\frac{dy}{dt} = ky$$ Rate of change is proportional to the amount present.

$$\frac{1}{y}dy = k\ dt$$ Divide both sides by y.

$$\int \frac{1}{y}dy = \int k\ dt$$ Integrate both sides.

$$\ln|y| = kt + C$$

→

$$\int \frac{1}{y}dy = \int k\ dt$$ Integrate both sides.

$$\ln|y| = kt + C$$

$$e^{\ln|y|} = e^{kt+C}$$ Exponentiate both sides.

$$|y| = e^C \cdot e^{kt}$$ When multiplying like bases, add exponents. So added exponents can be written as multiplication.

→

$$e^{\ln|y|} = e^{kt+C}$$ Exponentiate both sides.

$$|y| = e^C \cdot e^{kt}$$ When multiplying like bases, add exponents. So added exponents can be written as multiplication.

$$y = \pm e^C e^{kt}$$

$$y = Ae^{kt}$$ Since $\pm e^C$ is a constant, let $\pm e^C = A$.

→

$$y = \pm e^{C} e^{kt}$$

$$y = Ae^{kt}$$ Since $\pm e^{C}$ is a constant, let $\pm e^{C} = A$.

$$y_0 = A\cancelto{1}{e^{k\cdot 0}}$$ At $t = 0$, $y = y_0$.

$$y_0 = A$$

$$\boxed{y = y_0 e^{kt}}$$ This is the solution to our original initial value problem.

Exponential Change: $y = y_0 e^{kt}$

If the constant k is positive then the equation represents growth. If k is negative then the equation represents decay.

Note: This lecture will talk about exponential change formulas and where they come from. The problems in this section of the book mostly involve using those formulas. There are good examples in the book, which I will not repeat here.

Continuously Compounded Interest

If money is invested in a fixed-interest account where the interest is added to the account k times per year, the amount present after t years is:

$$A(t) = A_0\left(1+\frac{r}{k}\right)^{kt}$$

If the money is added back more frequently, you will make a little more money.

The best you can do is if the interest is added continuously.

Of course, the bank does not employ some clerk to continuously calculate your interest with an adding machine.

We could calculate: $\lim_{k\to\infty} A_0\left(1+\frac{r}{k}\right)^{kt}$

but we won't learn how to find this limit until chapter 8.

(The TI-89 can do it now if you would like to try it.)

Since the interest is proportional to the amount present, the equation becomes:

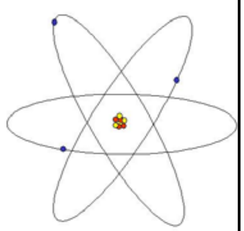
Continuously Compounded Interest:
$A = A_0 e^{rt}$

You may also use:

$A = Pe^{rt}$

which is the same thing.

Radioactive Decay



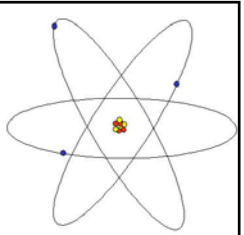
The equation for the amount of a radioactive element left after time *t* is:

$$y = y_0 e^{-kt}$$

This allows the decay constant, *k*, to be positive.

The *half-life* is the time required for half the material to decay.

Half-life



$$\frac{1}{2}\cancel{y}_0 = \cancel{y}_0 e^{-kt}$$

$$\ln\left(\frac{1}{2}\right) = \ln\left(e^{-kt}\right)$$

$$\overset{0}{\cancel{\ln 1}} - \ln 2 = -kt$$

$$\ln 2 = kt$$

$$\frac{\ln 2}{k} = t$$

Half-life:

$$\text{half-life} = \frac{\ln 2}{k}$$

Newton's Law of Cooling

Espresso left in a cup will cool to the temperature of the surrounding air. The rate of cooling is proportional to the difference in temperature between the liquid and the air.

(It is assumed that the air temperature is constant.)

If we solve the differential equation: $\frac{dT}{dt} = -k\left[T - T_s\right]$

we get:

Newton's Law of Cooling

$$T - T_s = \left[T_0 - T_s\right]e^{-kt}$$

where T_s is the temperature of the surrounding medium, which is a constant.

π

6.5 Partial Fractions

Objectives: •Evaluate integrals using partial fractions

1

$$\int \frac{5x-3}{x^2-2x-3}\,dx$$

→

2

$$\frac{6x+7}{(x+2)^2}$$

→

3

$$\frac{2x^3-4x^2-x-3}{x^2-2x-3}$$

→

4

Find the general solution to $\frac{dy}{dx}=\frac{6x^2-8x-4}{(x^2-4)(x-1)}$

→

6.5 Population Growth

Objectives:
- Group Presentations
- Solve problems involving exponential or logistic population grown.

We have used the exponential growth equation $y = y_0 e^{kt}$ to represent population growth.

The exponential growth equation occurs when the rate of growth is proportional to the amount present.

If we use P to represent the population, the differential equation becomes: $\frac{dP}{dt} = kP$

The constant k is called the relative growth rate.

$$\frac{dP/dt}{P} = k$$

→

The population growth model becomes: $P = P_0 e^{kt}$

However, real-life populations do not increase forever. There is some limiting factor such as food, living space or waste disposal.

There is a maximum population, or carrying capacity, M.

A more realistic model is the logistic growth model where growth rate is proportional to both the amount present (P) and the fraction of the carrying capacity that remains: $\frac{M-P}{M}$

→

The equation then becomes:

$$\frac{dP}{dt} = kP\left(\frac{M-P}{M}\right)$$

Our book writes it this way:

Logistics Differential Equation

$$\frac{dP}{dt} = \frac{k}{M}P(M-P)$$

We can solve this differential equation to find the logistics growth model.

→

Logistics Differential Equation

$$\frac{dP}{dt} = \frac{k}{M}P(M-P)$$

→

Logistics Growth Model

$$P = \frac{M}{1+Ae^{-kt}}$$

→

Example:

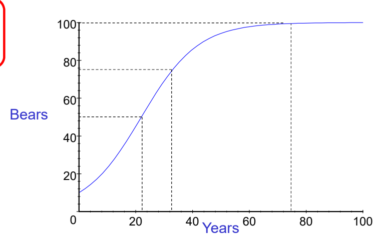
Logistic Growth Model

Ten grizzly bears were introduced to a national park 10 years ago. There are 23 bears in the park at the present time. The park can support a maximum of 100 bears.

Assuming a logistic growth model, when will the bear population reach 50? 75? 100?

$$\boxed{P = \frac{100}{1 + 9e^{-0.1t}}}$$

We can graph this equation and use "trace" to find the solutions.



y=50 at 22 years

y=75 at 33 years

y=100 at 75 years

Day 61/62
11/17/14

7.1 Integral as Net Change

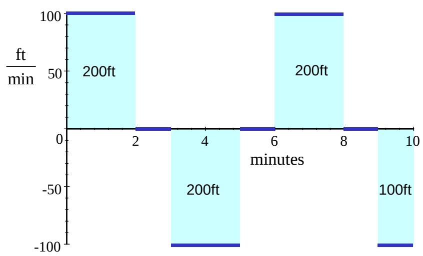
Objectives: •Solve problems in which a rate is integrated to find the net change over time in a variety of applications

Assignment: pg. 386 #'s 1-15 odd, 17-20, 31-36

A honey bee makes several trips from the hive to a flower garden. The velocity graph is shown below.

What is the total distance traveled by the bee?

$200 + 200 + 200 + 100 = 700$ 700 feet

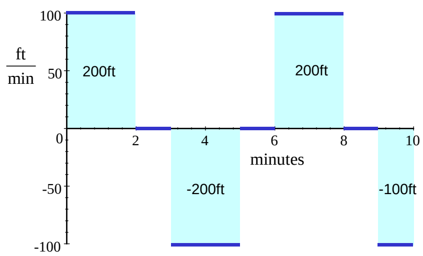


→

What is the displacement of the bee?

$200 - 200 + 200 - 100 = 100$

100 feet towards the hive



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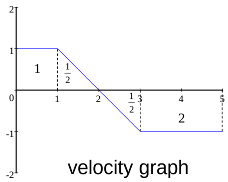
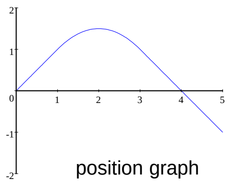


velocity graph



position graph

Displacement:

$$1+\frac{1}{2}-\frac{1}{2}-2=-1$$

Distance Traveled:

$$1+\frac{1}{2}+\frac{1}{2}+2=4$$

Every AP exam I have seen has had at least one problem requiring students to interpret velocity and position graphs.

→

To find the displacement (position shift) from the velocity function, we just integrate the function. The negative areas below the x-axis subtract from the total displacement.

$$\text{Displacement} = \int_a^b V(t)\,dt$$

To find distance traveled we have to use absolute value.

$$\text{Distance Traveled} = \int_a^b |V(t)|\,dt$$

Find the roots of the velocity equation and integrate in pieces, just like when we found the area between a curve and the x-axis. (Take the absolute value of each integral.)

Or you can use your calculator to integrate the absolute value of the velocity function. (However, on the AP exam, they look for the roots of the velocity equation)

→

An object has the following velocity $v(t)=t^2-\frac{8}{(t+1)^2}$

Find the displacement the object travels in the1st second

$$s(t)=\int_0^1 t^2-\frac{8}{(t+1)^2}dt$$

$$=\left.\frac{t^3}{3}+\frac{8}{t+1}\right|_0^1$$

$$=\frac{1}{3}+4-8=-\frac{11}{3}$$

An object has the following velocity $v(t)=t^2-\frac{8}{(t+1)^2}$

Find the distance the object travels in 2 seconds.

First you have to find when the object stops, i.e. when the velocity is zero.

$$0=t^2-\frac{8}{(t+1)^2} \qquad \cancel{t=-2.2545...} \qquad t=1.2545...$$

Then we need to find out when the object is moving in the positive direction and the negative direction

$$v(t) \quad - \quad 0 \quad +$$

$t=1.2545...$

An object has the following velocity $v(t)=t^2-\frac{8}{(t+1)^2}$

Now integrate in different pieces using the bounds when the object is stop. Do not forgot to take the absolute value integral when the object is moving to the left.

$$s(t)=\int_0^{1.2545}\left|t^2-\frac{8}{(t+1)^2}\right|dt+\int_0^2 t^2-\frac{8}{(t+1)^2}dt$$

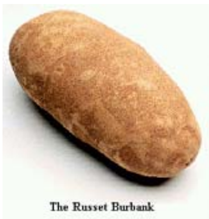
$$=4.9202$$

Example 5: National Potato Consumption

The rate of potato consumption for a particular country was:

$$C(t)=2.2+1.1^t$$

where t is the number of years since 1970 and C is in millions of bushels per year.



For a small Δt, the rate of consumption is constant.

The amount consumed during that short time is $C(t)\cdot\Delta t$.

→

Example 5: National Potato Consumption

$$C(t) = 2.2 + 1.1^t$$

The amount consumed during that short time is $C(t) \cdot \Delta t$.

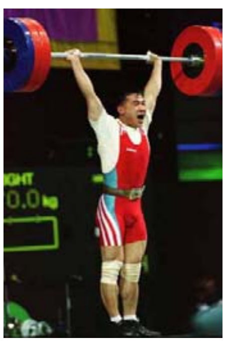
We add up all these small amounts to get the total consumption:

$$\text{total consumption} = \int C(t)\, dt$$

From the beginning of 1972 to the end of 1973:

$$\int_2^4 2.2 + 1.1^t dt \quad = 2.2t + \frac{1}{\ln 1.1} 1.1^t \Big|_2^4 \quad \approx 7.066$$ million bushels

→



Work:

$$\text{work} \;= \text{force} \cdot \text{distance}$$

Calculating the work is easy when the force and distance are constant.

When the amount of force varies, we get to use calculus!

→

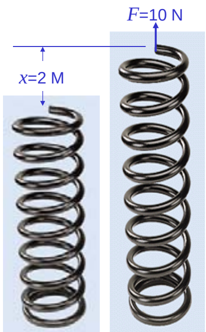
Hooke's law for springs: $F = kx$

k = spring constant

x = distance that the spring is extended beyond its natural length

→

Hooke's law for springs: $F = kx$



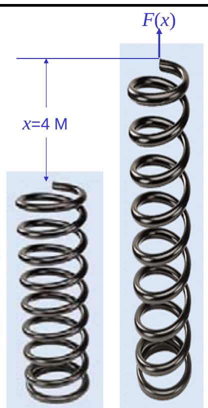
Example 7:

It takes 10 Newtons to stretch a spring 2 meters beyond its natural length.

$10 = k \cdot 2$

$5 = k \longrightarrow F = 5 \cdot x$

How much work is done stretching the spring to 4 meters beyond its natural length?



How much work is done stretching the spring to 4 meters beyond its natural length?

For a very small change in x, the force is constant.

$dw = F(x)\,dx \qquad F(x) = 5x$

$dw = 5x\ dx$

$\int dw = \int 5x\ dx$

$W = \int_0^4 5x\ dx$

$W = \frac{5}{2}x^2 \Big|_0^4$

$W = 40$ newton-meters

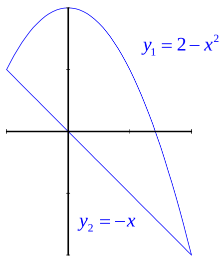
$W = 40$ joules

Day 63/64
11/19/14

7.2 Areas in the Plane

Objectives: •Use integration to calculate areas of regions in a plane

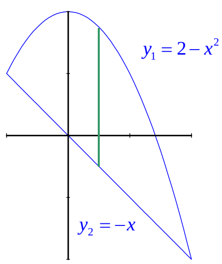
Assignment: pg. 395 #'s 2-42 even, 50-55



How can we find the area between these two curves?

We could split the area into several sections, use subtraction and figure it out, but there is an easier way.

→



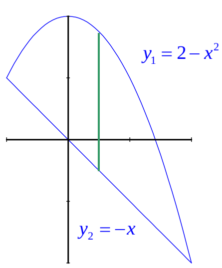
Consider a very thin vertical strip.

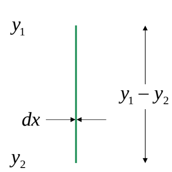
The length of the strip is:

$y_1 - y_2$ or $\left(2 - x^2\right) - \left(-x\right)$

Since the width of the strip is a very small change in x, we could call it dx.

→



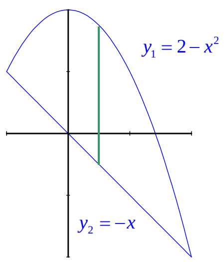


Since the strip is a long thin rectangle, the area of the strip is:

$$\text{length} \cdot \text{width} = \left(2 - x^2 + x\right) dx$$

If we add all the strips, we get: $\int_{-1}^{2} 2 - x^2 + x \, dx$

→



$$\int_{-1}^{2} 2 - x^2 + x \, dx$$

$$2x - \frac{1}{3}x^3 + \frac{1}{2}x^2 \Big|_{-1}^{2}$$

$$\left(4 - \frac{8}{3} + 2\right) - \left(-2 + \frac{1}{3} + \frac{1}{2}\right)$$

$$6 - \frac{8}{3} + 2 - \frac{1}{3} - \frac{1}{2}$$

$$\frac{36 - 16 + 12 - 2 - 3}{6} = \frac{27}{6} = \frac{9}{2}$$

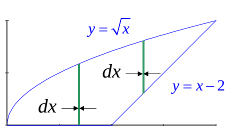
→

The formula for the area between curves is:

$$\text{Area} = \int_a^b \left[f_1(x) - f_2(x) \right] dx$$

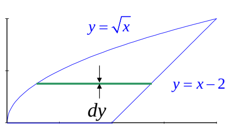
We will use this so much, that you won't need to "memorize" the formula!

→



If we try vertical strips, we have to integrate in two parts:

$$\int_0^2 \sqrt{x} \, dx + \int_2^4 \sqrt{x} - (x - 2) \, dx$$



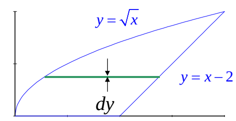
We can find the same area using a horizontal strip.

Since the width of the strip is dy, we find the length of the strip by solving for x in terms of y.

$y = \sqrt{x}$ $\quad y = x - 2$

$y^2 = x$ $\quad y + 2 = x$

→



We can find the same area using a horizontal strip.

Since the width of the strip is dy, we find the length of the strip by solving for x in terms of y.

$y=\sqrt{x}$ $\quad$ $y=x-2$

$y^2=x$ $\quad$ $y+2=x$

$$\int_0^2 \underbrace{(y+2)-y^2}_{\text{length of strip}}\ \underbrace{dy}_{\text{width of strip}}$$

$$\frac{1}{2}y^2+2y-\frac{1}{3}y^3\Big|_0^2$$

$$2+4-\frac{8}{3}=\frac{10}{3}$$

→

General Strategy for Area Between Curves:

1. Sketch the curves.
2. Decide on vertical or horizontal strips. (Pick whichever is easier to write formulas for the length of the strip, and/or whichever will let you integrate fewer times.)
3. Write an expression for the area of the strip. (If the width is dx, the length must be in terms of x. If the width is dy, the length must be in terms of y.)
4. Find the limits of integration. (If using dx, the limits are x values; if using dy, the limits are y values.)
5. Integrate to find area.

π

Day 63/64
11/19/14

7.3 Volumes

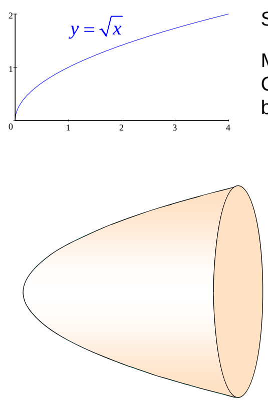
Objectives: •Use integration to calculate volumes of solids by cross sections
•Use integration to calculate surface areas of solids of revolutions

Assignment: pg. 405 Quick Review #'s 1-10,
pg. 406 #'s 1-14, 15-41 odd, 63-68,
AP Review #'s 1-3

Method of Slicing (p400):

1. Sketch the solid and a typical cross section.
2. Find a formula for $A(x)$.
3. Find the limits of integration.
4. Integrate $A(x)$ to find volume, $V(x)$

→

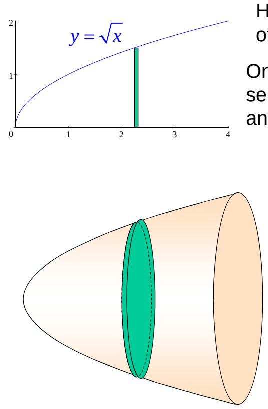


Suppose I start with this curve.

My boss at the ACME Rocket Company has assigned me to build a nose cone in this shape.

So I put a piece of wood in a lathe and turn it to a shape to match the curve.

→



How could we find the volume of the cone?

One way would be to cut it into a series of thin slices (flat cylinders) and add their volumes.

The volume of each flat cylinder (disk) is:

$\pi r^2 \cdot$ the thickness

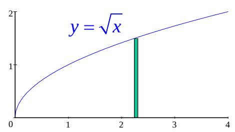
$\pi \left(\sqrt{x}\right)^2 \; dx$

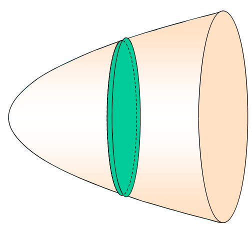
In this case:

$r =$ the y value of the function

thickness $=$ a small change in $x = dx$

→





The volume of each flat cylinder (disk) is:

$\pi r^2 \cdot$ the thickness

$\pi \left(\sqrt{x}\right)^2 \; dx$

If we add the volumes, we get:

$$\int_0^4 \pi \left(\sqrt{x}\right)^2 dx$$

$$= \int_0^4 \pi x \, dx$$

$$= \frac{\pi}{2} x^2 \Big|_0^4 \qquad = 8\pi$$

This application of the method of slicing is called the <u>disk method</u>. The shape of the slice is a disk, so we use the formula for the area of a circle to find the volume of the disk.

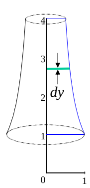
If the shape is rotated about the x-axis, then the formula is:

$$V = \pi \int_a^b y^2 \; dx$$

A shape rotated about the y-axis would be: $V = \pi \int_a^b x^2 \; dy$

The region between the curve $x = \frac{1}{\sqrt{y}}$, $1 \le y \le 4$ and the *y*-axis is revolved about the <u>*y*-axis</u>. Find the volume.

y	x
1	1
2	$\frac{1}{\sqrt{2}} = .707$
3	$\frac{1}{\sqrt{3}} = .577$
4	$\frac{1}{2}$



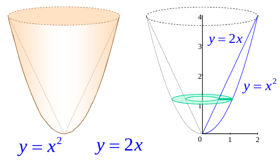
We use a horizontal disk.

The thickness is dy.

The radius is the x value of the function $= \frac{1}{\sqrt{y}}$.

$$V = \underbrace{\int_1^4 \pi \left(\frac{1}{\sqrt{y}}\right)^2 dy}_{\text{volume of disk}} \qquad = \int_1^4 \pi \frac{1}{y} \, dy$$

$$= \pi \ln y \Big|_1^4 \quad = \pi \left(\ln 4 - \cancelto{0}{\ln 1}\right) \quad = \pi \ln 2^2 \quad = 2\pi \ln 2$$



$y=x^2$ $y=2x$

$\sqrt{y}=x$ $\frac{y}{2}=x$

The region bounded by $y=x^2$ and $y=2x$ is revolved about the y-axis. Find the volume.

If we use a horizontal slice:

The "disk" now has a hole in it, making it a "washer".

The volume of the washer is: $\left(\pi R^2-\pi r^2\right)\cdot\text{thickness}$

$$V=\int_0^4\pi\left(\left(\sqrt{y}\right)^2-\left(\frac{y}{2}\right)^2\right)dy$$

$$\pi\left(R^2-r^2\right)dy$$

outer radius inner radius

$$V=\int_0^4\pi\left(y-\frac{1}{4}y^2\right)dy$$

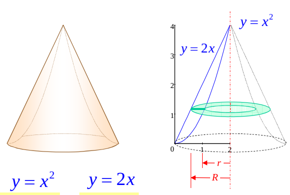
$$V=\pi\int_0^4 y-\frac{1}{4}y^2\ dy \quad =\pi\left[\frac{1}{2}y^2-\frac{1}{12}y^3\right]_0^4 \quad =\pi\left[8-\frac{16}{3}\right] \quad =\frac{8\pi}{3}$$

→

This application of the method of slicing is called the washer method. The shape of the slice is a circle with a hole in it, so we subtract the area of the inner circle from the area of the outer circle.

The washer method formula is: $V=\pi\int_a^b R^2-r^2\ dx$

→



$y=x^2$ $y=2x$

$\sqrt{y}=x$ $\frac{y}{2}=x$

If the same region is rotated about the line $x=2$:

The outer radius is:

$$R=2-\frac{y}{2}$$

The inner radius is:

$$r=2-\sqrt{y}$$

$$V=\pi\int_0^4 R^2-r^2dy$$

$$=\pi\int_0^4\left(2-\frac{y}{2}\right)^2-\left(2-\sqrt{y}\right)^2dy$$

$$=\pi\int_0^4\left(4-2y+\frac{y^2}{4}\right)-\left(4-4\sqrt{y}+y\right)dy$$

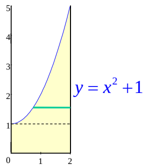
$$=\pi\int_0^4\cancel{4}-2y+\frac{y^2}{4}-\cancel{4}+4\sqrt{y}-y\ dy$$

$$=\pi\int_0^4-3y+\frac{1}{4}y^2+4y^{\frac{1}{2}}\ dy$$

$$=\pi\cdot\left[-\frac{3}{2}y^2+\frac{1}{12}y^3+\frac{8}{3}y^{\frac{3}{2}}\right]_0^4$$

$$=\pi\cdot\left[-24+\frac{16}{3}+\frac{64}{3}\right] \quad =\frac{8\pi}{3}$$

→



$$\pi\int_1^5 4-(y-1)\,dy+4\pi$$

$$\pi\int_1^5 5-y\,dy+4\pi$$

$$\pi\left[5y-\frac{1}{2}y^2\right]_1^5+4\pi$$

$$\pi\left[\left(25-\frac{25}{2}\right)-\left(5-\frac{1}{2}\right)\right]+4\pi$$

$$\pi\left[\frac{25}{2}-\frac{9}{2}\right]+4\pi$$

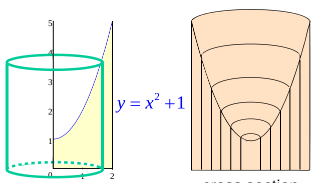
$$\pi\cdot\frac{16}{2}+4\pi$$

$$8\pi+4\pi \qquad =12\pi$$

$y-1=x^2 \qquad x=\sqrt{y-1}$

$$\pi\int_1^5 \underset{\text{outer radius}}{2^2}-\underbrace{\left(\sqrt{y-1}\right)^2}_{\text{inner radius}}\underset{\text{thickness of slice}}{dy}+\underbrace{\pi\cdot 2^2\cdot 1}_{\text{cylinder}}$$

→



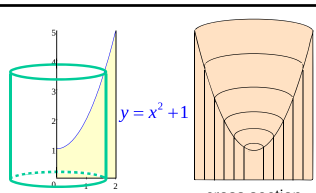
cross section

Here is another way we could approach this problem:

If we take a vertical slice and revolve it about the y-axis we get a cylinder.

If we add all of the cylinders together, we can reconstruct the original object.

→



cross section

The volume of a thin, hollow cylinder is given by:

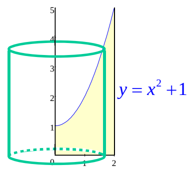
Lateral surface area of cylinder $\cdot$ thickness

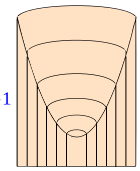
$=$ circumference $\cdot$ height $\cdot$ thickness

$=2\pi r\cdot h\cdot$ thickness

$$=\underbrace{2\pi \underset{r}{x}}_{\text{circumference}}\underbrace{\left(x^2+1\right)}_{h}\underset{\text{thickness}}{dx}$$

→





cross section

This is called the <u>shell method</u> because we use cylindrical shells.

If we add all the cylinders from the smallest to the largest:

$$\int_0^2 2\pi x\left(x^2+1\right)\,dx$$

$$=2\pi r \cdot h \cdot \text{thickness}$$

$$=\underbrace{2\pi \underbrace{x}_{r}}_{\text{circumference}}\underbrace{\left(x^2+1\right)}_{h}\,\underbrace{dx}_{\text{thickness}}$$

$$2\pi\int_0^2 x^3 + x\,dx$$

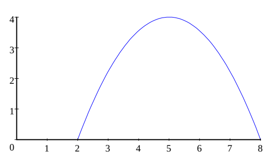
$$2\pi\left[\frac{1}{4}x^4 + \frac{1}{2}x^2\right]_0^2$$

$$2\pi\left[4+2\right]$$

$$12\pi$$

→

Find the volume generated when this shape is revolved about the y axis.

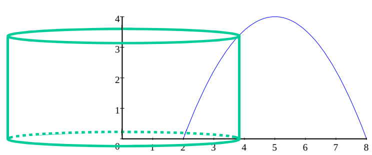


$$y = -\frac{4}{9}\left(x^2 - 10x + 16\right)$$

We can't solve for x, so we can't use a horizontal slice directly.

→

If we take a vertical slice and revolve it about the y-axis we get a cylinder.



$$y = -\frac{4}{9}\left(x^2 - 10x + 16\right)$$

Shell method:

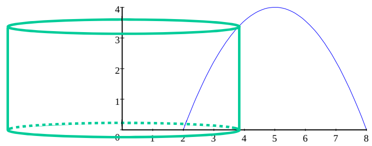
Lateral surface area of cylinder

$=\text{circumference} \cdot \text{height}$

$=2\pi r \cdot h$

Volume of thin cylinder $= 2\pi r \cdot h \cdot dx$

→



Volume of thin cylinder $= 2\pi r \cdot h \cdot dx$

$y = -\frac{4}{9}\left(x^2 - 10x + 16\right)$

$$\int_2^8 \underbrace{2\pi \overset{r}{x}}_{\text{circumference}} \underbrace{\left[-\frac{4}{9}\left(x^2 - 10x + 16\right)\right]}_{h} \underbrace{dx}_{\text{thickness}}$$

$= 160\pi$

$\approx 502.655 \text{ cm}^3$

Note: When entering this into the calculator, be sure to enter the multiplication symbol before the parenthesis.

→

When the strip is parallel to the axis of rotation, use the shell method.

When the strip is perpendicular to the axis of rotation, use the washer method.

π

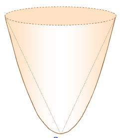
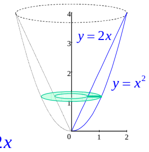
Day 71/72
12/4/13

7.3 Volumes

Objectives: •Use integration to calculate volumes of solids of revolutions

Assignment: pg. 407 #'s 15-41 odd, 63-68,
AP Review #'s 1-3

Quiz 7.1-7.3 Monday
Free Response Tuesday

The region bounded by $y=x^2$ and $y=2x$ is revolved about the y-axis. Find the volume.

If we use a horizontal slice:

The "disk" now has a hole in it, making it a "washer".

$y=x^2$ $\quad y=2x$

$\sqrt{y}=x$ $\quad \frac{y}{2}=x$

The volume of the washer is: $\left(\pi R^2-\pi r^2\right)\cdot \text{thickness}$

$$V=\int_0^4 \pi\left(\left(\sqrt{y}\right)^2-\left(\frac{y}{2}\right)^2\right)dy$$

$$\pi\left(R^2-r^2\right)dy$$

outer radius $\quad$ inner radius

$$V=\int_0^4 \pi\left(y-\frac{1}{4}y^2\right)dy$$

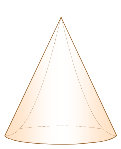
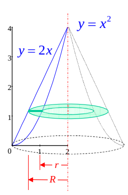
$$V=\pi\int_0^4 y-\frac{1}{4}y^2\ dy \quad =\pi\left[\frac{1}{2}y^2-\frac{1}{12}y^3\right]_0^4 \quad =\pi\left[8-\frac{16}{3}\right] \quad =\frac{8\pi}{3}$$

→

This application of the method of slicing is called the washer method. The shape of the slice is a circle with a hole in it, so we subtract the area of the inner circle from the area of the outer circle.

The washer method formula is: $V=\pi\int_a^b R^2-r^2\ dx$

→

If the same region is rotated about the line $x=2$:

The outer radius is:

$$R=2-\frac{y}{2}$$

The inner radius is:

$$r=2-\sqrt{y}$$

$y=x^2$ $\quad y=2x$

$\sqrt{y}=x$ $\quad \frac{y}{2}=x$

$$V=\pi\int_0^4 R^2-r^2 dy$$

$$=\pi\int_0^4\left(2-\frac{y}{2}\right)^2-\left(2-\sqrt{y}\right)^2 dy$$

$$=\pi\int_0^4\left(4-2y+\frac{y^2}{4}\right)-\left(4-4\sqrt{y}+y\right)dy$$

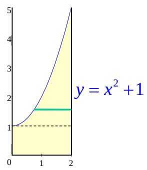
$$=\pi\int_0^4 \cancel{4}-2y+\frac{y^2}{4}-\cancel{4}+4\sqrt{y}-y\ dy$$

$$=\pi\int_0^4 -3y+\frac{1}{4}y^2+4y^{\frac{1}{2}}\ dy$$

$$=\pi\cdot\left[-\frac{3}{2}y^2+\frac{1}{12}y^3+\frac{8}{3}y^{\frac{3}{2}}\right]_0^4$$

$$=\pi\cdot\left[-24+\frac{16}{3}+\frac{64}{3}\right] \quad =\frac{8\pi}{3}$$

→



$y-1=x^2 \qquad x=\sqrt{y-1}$

$$\pi\int_1^5 \underbrace{2^2}_{\text{outer radius}}-\underbrace{\left(\sqrt{y-1}\right)^2}_{\text{inner radius}}\underbrace{dy}_{\text{thickness of slice}}+\underbrace{\pi\cdot 2^2\cdot 1}_{\text{cylinder}}$$

$$\pi\int_1^5 4-(y-1)\,dy+4\pi$$

$$\pi\int_1^5 5-y\,dy+4\pi$$

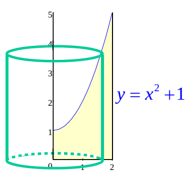
$$\pi\left[5y-\frac{1}{2}y^2\right]_1^5+4\pi$$

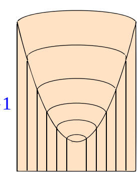
$$\pi\left[\left(25-\frac{25}{2}\right)-\left(5-\frac{1}{2}\right)\right]+4\pi$$

$$\pi\left[\frac{25}{2}-\frac{9}{2}\right]+4\pi$$

$$\pi\cdot\frac{16}{2}+4\pi$$

$$8\pi+4\pi \qquad =12\pi$$



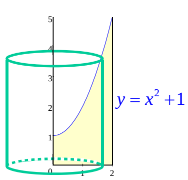


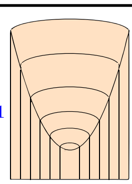
cross section

Here is another way we could approach this problem:

If we take a vertical slice and revolve it about the y-axis we get a cylinder.

If we add all of the cylinders together, we can reconstruct the original object.





cross section

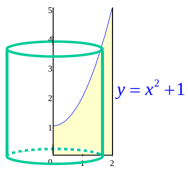
The volume of a thin, hollow cylinder is given by:

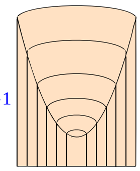
Lateral surface area of cylinder $\cdot$ thickness

$=$ circumference $\cdot$ height $\cdot$ thickness

$=2\pi r\cdot h\cdot$ thickness

$$=\underbrace{2\pi \underset{r}{x}}_{\text{circumference}}\underbrace{\left(x^2+1\right)}_{h}\underset{\text{thickness}}{dx}$$





cross section

This is called the shell method because we use cylindrical shells.

If we add all the cylinders from the smallest to the largest:

$$\int_0^2 2\pi x\left(x^2+1\right)\,dx$$

$$=2\pi r\cdot h\cdot \text{thickness}$$

$$=\underbrace{2\pi \overset{\uparrow}{\underset{r}{x}}}_{\text{circumference}}\underbrace{\left(x^2+1\right)}_{h}\;\underbrace{dx}_{\text{thickness}}$$

$$2\pi\int_0^2 x^3+x\,dx$$

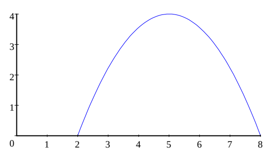
$$2\pi\left[\frac{1}{4}x^4+\frac{1}{2}x^2\right]_0^2$$

$$2\pi\left[4+2\right]$$

$$12\pi$$

→

Find the volume generated when this shape is revolved about the y axis.

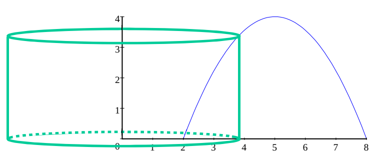


$$y=-\frac{4}{9}\left(x^2-10x+16\right)$$

We can't solve for x, so we can't use a horizontal slice directly.

→

If we take a vertical slice and revolve it about the y-axis we get a cylinder.



$$y=-\frac{4}{9}\left(x^2-10x+16\right)$$

Shell method:

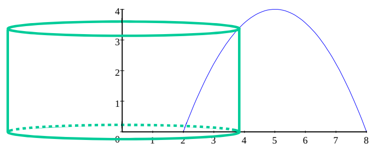
Lateral surface area of cylinder

$$=\text{circumference}\cdot\text{height}$$

$$=2\pi r\cdot h$$

$$\text{Volume of thin cylinder}=2\pi r\cdot h\cdot dx$$

→



Volume of thin cylinder $= 2\pi r \cdot h \cdot dx$

$y = -\frac{4}{9}\left(x^2 - 10x + 16\right)$

$$\int_2^8 \underbrace{2\pi \underbrace{x}_{r}}_{\text{circumference}} \underbrace{\left[-\frac{4}{9}\left(x^2 - 10x + 16\right)\right]}_{h} \underbrace{dx}_{\text{thickness}}$$

$= 160\pi$

$\approx 502.655 \text{ cm}^3$

Note: When entering this into the calculator, be sure to enter the multiplication symbol before the parenthesis.

When the strip is parallel to the axis of rotation, use the shell method.

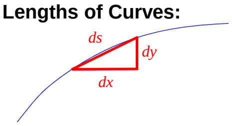
When the strip is perpendicular to the axis of rotation, use the washer method.

7.4 Lengths of Curves

Objectives: •Use integration to calculate lengths of curves in a plane

Assignment: pg. 416 #'s 1-18, 32-37

Lengths of Curves:



If we want to approximate the length of a curve, over a short distance we could measure a straight line.

By the pythagorean theorem:

$$ds^2 = dx^2 + dy^2$$

$$ds = \sqrt{dx^2 + dy^2}$$

$$\int ds = \int \sqrt{dx^2 + dy^2}$$

$$S = \int \sqrt{\left(\frac{dx^2}{dx^2} + \frac{dy^2}{dx^2}\right) dx^2}$$

$$L = \int \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}\ dx$$

Length of Curve (Cartesian)

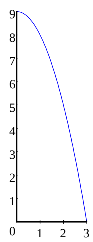
$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

→

Example:

$y = -x^2 + 9$

$0 \le x \le 3$



$$y = -x^2 + 9$$

$$\frac{dy}{dx} = -2x$$

$$L = \int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$$L = \int_0^3 \sqrt{1 + (-2x)^2}\ dx$$

$$L = \int_0^3 \sqrt{1 + 4x^2}\ dx$$

Now what? This doesn't fit any formula, and we started with a pretty simple example!

The TI-89 gets:

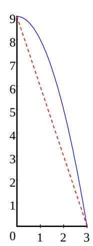
$$L = \frac{\ln\left(\sqrt{37} + 6\right)}{4} + \frac{3\sqrt{37}}{2} \qquad \approx 9.74708875861$$

→

Example:

$y = -x^2 + 9$

$0 \le x \le 3$



If we check the length of a straight line:

$$9^2 + 3^2 = C^2$$

$$81 + 9 = C^2$$

$$90 = C^2$$

$$C \approx 9.49$$

The curve should be a little longer than the straight line, so our answer seems reasonable.

The TI-89 gets:

$$L = \frac{\ln\left(\sqrt{37} + 6\right)}{4} + \frac{3\sqrt{37}}{2} \qquad \approx 9.74708875861$$

→

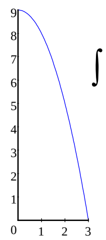
Example:

$y=-x^2+9$

$0 \le x \le 3$

You may want to let the calculator find the derivative too:

$(-)x \wedge 2+9$ STO→ Y ENTER



$\int(\sqrt{\ }(1+d(y,x)\wedge 2),x,0,3)$ ◆ ENTER

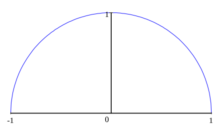
$$\int_0^3 \sqrt{1+\left(\frac{d}{dx}(y)\right)^2}\,dx$$

Important:
You must delete the variable *y* when you are done!

F4 4 Y ENTER

→

Example:



$$x^2+y^2=1$$

$$y^2=1-x^2$$

$$y=\sqrt{1-x^2}$$

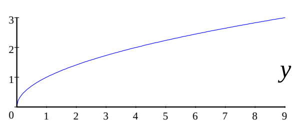
$$L=\int_{-1}^{1}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\approx 3.1415926536$$

$$\approx \pi$$

→

If you have an equation that is easier to solve for x than for y, the length of the curve can be found the same way.



$x=y^2 \qquad 0 \le y \le 3$

$y\wedge 2$ STO→ X ENTER

$$L=\int_0^3\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

Notice that *x* and *y* are reversed.

$\int(\sqrt{\ }(1+d(x,y)\wedge 2),y,0,3) \quad \approx 9.74708875861$

→

Don't forget to clear the x and y variables when you are done!

F4 4 X , Y ENTER

π

Day 79
12/16/13

7.5 Applications from Science and Stats

Objectives: •Model problems involving rates of change in a variety of applications

Assignment: pg. 425 #'s 1-6, 17, 36-39,
AP Review #'s 1-3

Chapter 7 Test- Tuesday

Review: Hooke's Law: $F = kx$

A spring has a natural length of 1 m.
A force of 24 N stretches the spring to 1.8 m.

a Find k: $F = kx$

$$24 = k(.8)$$

$$30 = k \qquad F = 30x$$

b How much work would be needed to stretch the spring 3m beyond its natural length?

$$W = \int_a^b F(x)\,dx \qquad W = 15x^2\Big|_0^3$$

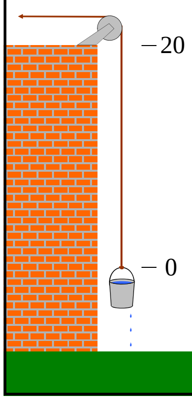
$$W = \int_0^3 30x\,dx \qquad W = 135 \text{ newton-meters}$$

→

Over a very short distance, even a non-constant force doesn't change much, so work becomes: $F(x)\,dx$

If we add up all these small bits of work we get:

$$W=\int_a^b F(x)\,dx$$



A leaky 5 lb bucket is raised 20 feet

The rope weights 0.08 lb/ft.

The bucket starts with 2 gal (16 lb) of water and is empty when it just reaches the top.

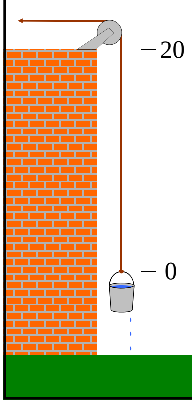
Work:

Bucket: $5\text{ lb}\cdot 20\text{ ft}=100\text{ ft-lb}$

Water: The force is proportional to remaining rope.

$$F(x)=\frac{20-x}{5\,\cancel{20}}\cdot\overset{4}{\cancel{16}}=16-\frac{4}{5}x$$

$$W=\int_a^b F(x)\,dx=\int_0^{20}16-\frac{4}{5}x\,dx$$



A leaky 5 lb bucket is raised 20 feet

The rope weights 0.08 lb/ft.

The bucket starts with 2 gal (16 lb) of water and is empty when it just reaches the top.

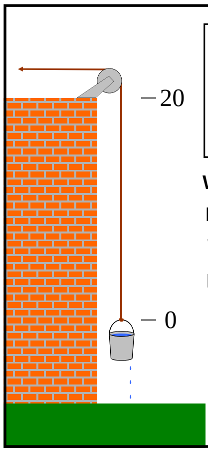
Work:

Bucket: $5\text{ lb}\cdot 20\text{ ft}=100\text{ ft-lb}$

Water: $W=\int_0^{20}16-\frac{4}{5}x\,dx$

$$W=16x-\frac{2}{5}x^2\Big|_0^{20}$$

$$W=16\cdot 20-\frac{2\cdot 20^2}{5}=160\text{ ft-lb}$$



A leaky 5 lb bucket is raised 20 feet

The rope weights 0.08 lb/ft.

The bucket starts with 2 gal (16 lb) of water and is empty when it just reaches the top.

Work:

Bucket: $5 \text{ lb} \cdot 20 \text{ ft} = 100 \text{ ft-lb}$

Water: $W = 160 \text{ ft-lb}$

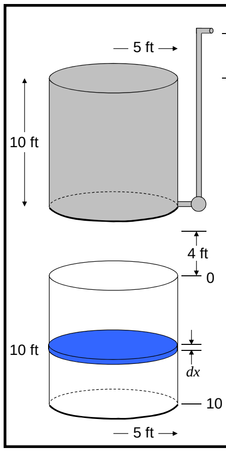
Rope: $F(x) = (20 - x)(0.08)$

$$W = \int_0^{20} (1.6 - .08x)\, dx$$

$$W = 1.6x - .04x^2 \Big|_0^{20} = 16 \text{ ft-lb}$$

Total: $100 + 160 + 16 = \boxed{276 \text{ ft-lb}}$

→



I want to pump the water out of this tank. How much work is done?

$$w = Fd$$

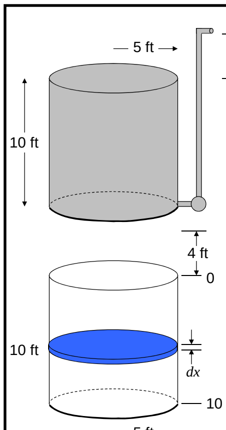
The force is the weight of the water. The water at the bottom of the tank must be moved further than the water at the top.

Consider the work to move one "slab" of water: Pg. 421

$$\text{weight of slab} = \text{density} \cdot \text{volume}$$
$$= 62.5 \cdot \pi \cdot 5^2\, dx$$
$$= 1562.5\pi\, dx$$

$$\text{distance} = x + 4$$

→



I want to pump the water out of this tank. How much work is done?

$$w = Fd$$

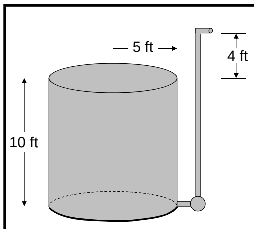
$$\text{weight of slab} = \text{density} \cdot \text{volume}$$
$$= 62.5 \cdot \pi \cdot 5^2\, dx$$
$$= 1562.5\pi\, dx$$

$$\text{distance} = x + 4$$

$$\text{work} = \underbrace{(x+4)}_{\text{distance}}\underbrace{1562.5\pi\, dx}_{\text{force}}$$

$$W = \int_0^{10} (x+4)1562.5\pi\, dx$$

→



I want to pump the water out of this tank. How much work is done?

$$\text{work} = \underbrace{(x+4)}_{\text{distance}}\underbrace{1562.5\pi\ dx}_{\text{force}}$$

$$W = \int_0^{10} (x+4)1562.5\pi\ dx$$

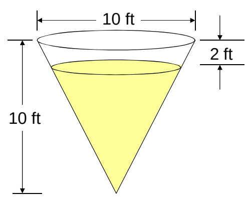
$$W = 1562.5\pi\left[\frac{1}{2}x^2 + 4x\right]_0^{10}$$

$$W = 1562.5\pi[50+40]$$

$$\boxed{W \approx 441{,}786 \text{ ft-lb}}$$

A 1 horsepower pump, rated at 550 ft-lb/sec, could empty the tank in just under 14 minutes!

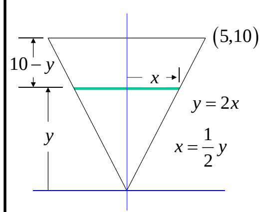
→



A conical tank is filled to within 2 ft of the top with salad oil weighing 57 lb/ft^3.

How much work is required to pump the oil to the rim?

Consider one slice (slab) first:



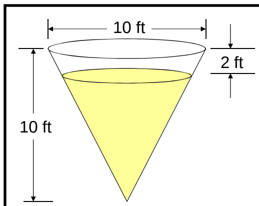
$$W_{(y)} = F \cdot d$$

$$W_{(y)} = \text{density} \cdot \text{volume} \cdot \text{distance}$$

$$W_{(y)} = 57\left[\pi\left(\frac{1}{2}y\right)^2\right]dy(10-y)$$

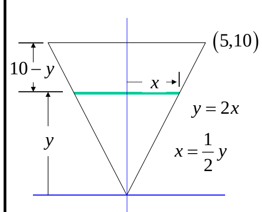
$$W = \int_0^8 (10-y)57\pi \cdot \frac{1}{4}y^2 dy$$

→



A conical tank if filled to within 2 ft of the top with salad oil weighing 57 lb/ft^3.

How much work is required to pump the oil to the rim?



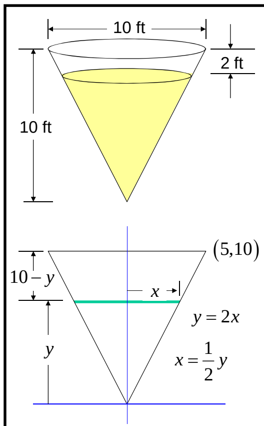
$$W_{(y)} = 57\left[\pi\left(\frac{1}{2}y\right)^2\right]dy(10-y)$$

$$W = \int_0^8 (10-y)57\pi \cdot \frac{1}{4}y^2 dy$$

$$W = \frac{57\pi}{4}\int_0^8 10y^2 - y^3\ dy$$

$$W = \frac{57\pi}{4}\left[\frac{10}{3}y^3 - \frac{y^4}{4}\right]_0^8$$

→



A conical tank if filled to within 2 ft of the top with salad oil weighing 57 lb/ft^3.

How much work is required to pump the oil to the rim?

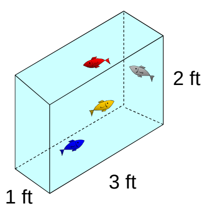
$$W = \frac{57\pi}{4}\int_0^8 10y^2 - y^3\ dy$$

$$W = \frac{57\pi}{4}\left[\frac{10}{3}y^3 - \frac{y^4}{4}\right]_0^8$$

$$W = \frac{57\pi}{4}\left[\frac{5120}{3} - \frac{4096}{4}\right]$$

$$\boxed{W \approx 30{,}561 \text{ ft-lb}}$$

π



What is the force on the bottom of the aquarium?

$$\begin{aligned} \text{Force} &= \text{weight of water} \\ &= \text{density} \cdot \text{volume} \\ &= 62.5\ \frac{\text{lb}}{\text{ft}^3} \cdot 2 \text{ ft} \cdot 3 \text{ ft} \cdot 1 \text{ ft} \\ &= 375 \text{ lb} \end{aligned}$$

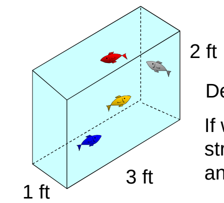
→

If we had a 1 ft x 3 ft plate on the bottom of a 2 ft deep wading pool, the force on the plate is equal to the weight of the water above the plate.

$$\underbrace{\underbrace{62.5\ \frac{\text{lb}}{\text{ft}^3}}_{\text{density}} \cdot \underbrace{2 \text{ ft}}_{\text{depth}}}_{\text{pressure}} \cdot \underbrace{3 \text{ ft} \cdot 1 \text{ ft}}_{\text{area}} = 375 \text{ lb}$$

All the other water in the pool doesn't affect the answer!

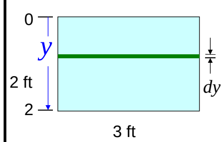
→



What is the force on the front face of the aquarium?

Depth (and pressure) are not constant.

If we consider a very thin horizontal strip, the depth doesn't change much, and neither does the pressure.



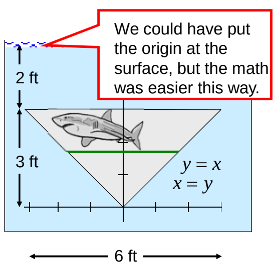
$$F_y = \underbrace{62.5}_{\text{density}} \cdot \underbrace{y}_{\text{depth}} \cdot \underbrace{3\,dy}_{\text{area}}$$

$$F = \int_0^2 62.5 \cdot y \cdot 3\,dy$$

$$F = \frac{187.5}{2} y^2 \Big|_0^2 = 375 \text{ lb}$$

It is just a coincidence that this matches the first answer!

→



A flat plate is submerged vertically as shown. (It is a window in the shark pool at the city aquarium.)

Find the force on one side of the plate.

Depth of strip: $(5-y)$

Length of strip: $2x = 2y$

Area of strip: $2y\,dy$

$$F_y = \underbrace{62.5}_{\text{density}}\underbrace{(5-y)}_{\text{depth}}\underbrace{2y\,dy}_{\text{area}}$$

$$F = \int_0^3 62.5(5-y)2y\,dy$$

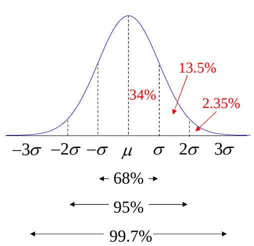
$$F = 125\int_0^3 5y - y^2\,dy$$

$$F = 125\left[\frac{5}{2}y^2 - \frac{1}{3}y^3\right]_0^3$$

$$\boxed{F = 1687.5 \text{ lb}}$$

→

Normal Distribution:



"68, 95, 99.7 rule"

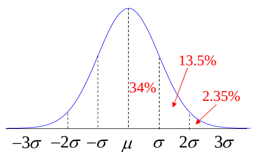
For many real-life events, a frequency distribution plot appears in the shape of a "normal curve".

The mean μ (or $\bar{x}$) is in the middle of the curve. The shape of the curve is determined by the standard deviation σ.

μ	mu
$\bar{x}$	x-bar
σ	sigma

→

Normal Distribution:



"68, 95, 99.7 rule"

The area under the curve from *a* to *b* represents the probability of an event occurring within that range.

In stat we used z-scores and a table of values to determine probabilities. If we know the equation of the curve we can use calculus (and our calculator) to determine probabilities:

Normal Probability Density Function: (Gaussian curve)

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\left(-(x-\mu)^2/2\sigma^2\right)}$$

Normal Distribution:

The good news is that you do <u>not</u> have to memorize this equation!

Example 7 on page 424 shows how you could integrate this function to predict probabilities.

In real life, statisticians rarely see this function. They use computer programs or graphing calculators with statistics software to draw the curve or predict the probabilities.

Normal Probability Density Function: (Gaussian curve)

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\left(-(x-\mu)^2/2\sigma^2\right)}$$