8.2A Quadratic Formula

A. Derivation of the Quadratic Formula

We can get a general formula for the solutions to $ax^2 + bx + c = 0$ by doing completing the square on the general equation.

$$ax^2 + bx + c = 0$$

$$a\left(x^2 + \frac{b}{a}x\right) + c = 0 \qquad \text{[Factor out, first two]}$$

$$a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right] + c - a\left(\frac{b}{2a}\right)^2 = 0 \quad \text{[Completing the square]}$$

$$a\left(x + \frac{b}{2a}\right)^2 + c - a\left(\frac{b^2}{4a^2}\right) = 0$$

$$a\left(x + \frac{b}{2a}\right)^2 = a\left(\frac{b^2}{4a^2}\right) - c$$

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a} - c$$

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a} - \frac{4ac}{4a}$$

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Quadratic Formula: $\boxed{x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$

B. Using the Quadratic Formula

Given $ax^2 + bx + c = 0$, we have $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Example 1: Solve $3x^2 + 5x + 1 = 0$ for x by the Quadratic Formula.

Solution

$a = 3, \quad b = 5, \quad c = 1$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-5 \pm \sqrt{(5)^2 - 4(3)(1)}}{2(3)} \\ &= \frac{-5 \pm \sqrt{25 - 12}}{6} = \frac{-5 \pm \sqrt{13}}{6} \end{aligned}$$

Ans $\boxed{\dfrac{-5 + \sqrt{13}}{6} \quad \text{or} \quad \dfrac{-5 - \sqrt{13}}{6}}$

Example 2: Solve $6x^2 - 2x - 1 = 0$ for x by the Quadratic Formula.

Solution

$a = 6, \quad b = -2, \quad c = -1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(6)(-1)}}{2(6)}$$

$$= \frac{2 \pm \sqrt{4 + 24}}{12} = \frac{2 \pm \sqrt{28}}{12}$$

Now simplify:

$$x = \frac{2 \pm \sqrt{28}}{12} = \frac{2 \pm 2\sqrt{7}}{12} = \frac{2(1 \pm \sqrt{7})}{12} = \frac{1 \pm \sqrt{7}}{6}$$

Ans $\boxed{\frac{1 + \sqrt{7}}{6} \quad \text{or} \quad \frac{1 - \sqrt{7}}{6}}$

Example 3: Solve $\dfrac{5x}{x-2} + \dfrac{6}{(x-2)(x+3)} = \dfrac{x-1}{x+3}$ for x

Solution

Disallowed values: $x \neq 2, -3$
LCD= $(x-2)(x+3)$
Multiply both sides by LCD:

$$(x-2)(x+3)\left[\frac{5x}{x-2} + \frac{6}{(x-2)(x+3)}\right] = (x-2)(x+3)\left(\frac{x-1}{x+3}\right)$$

$$5x(x+3) + 6 = (x-2)(x-1)$$

$$5x^2 + 15x + 6 = x^2 - 3x + 2$$

Move everything to one side:

$$4x^2 + 18x + 4 = 0$$

$$2(2x^2 + 9x + 2) = 0$$

Divide by 2:

$$2x^2 + 9x + 2 = 0$$

Now use the quadratic formula:

$$a = 2, \quad b = 9, \quad c = 2$$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-9 \pm \sqrt{(9)^2 - 4(2)(2)}}{2(2)} \\ &= \frac{-9 \pm \sqrt{81 - 16}}{4} = \frac{-9 \pm \sqrt{65}}{4} \end{aligned}$$

Neither of the solutions are disallowed.

Ans $\boxed{\dfrac{-9 + \sqrt{65}}{4} \quad \text{or} \quad \dfrac{-9 - \sqrt{65}}{4}}$