8.1C Completing the Square

A. Introduction

By the square formula, $(x-4)^2 = x^2 - 8x + 16$.

Suppose you knew that $x^2 - 8x$ were the first two terms of a perfect square, how could you figure out that the last term had to be 16?

Note: "half of -8 squared is 16"

Similarly, if you had x^2-10x, you need $(-\frac{10}{2})^2 = 25$ to get a perfect square of $x^2-10x+25$

B. Practice

Example 1: $x^2 + 6x$ are the first two terms of a perfect square. What is it?

Solution

$$(\frac{6}{2})^2 = 3^2 = 9$$

Ans $\boxed{x^2 + 6x + 9}$

Example 2: $x^2 - 14x$ are the first two terms of a perfect square. What is it?

Solution

$$(-\frac{14}{2})^2 = (-7)^2 = 49$$

Ans $\boxed{x^2 - 14x + 49}$

C. Completing the Square–Part 1

Making perfect square trinomials as above is called **completing the square**.

In general, though, we can't just add on the extra number, because that would change the problem.

To keep things equal, we must subtract it off.

We write the answer in factored form.

D. Examples–Part 1

Example 1: Complete the square on $x^2 + 12x$.

Solution

$(\frac{12}{2})^2 = 6^2 = 36$

Thus, we have

$x^2 + 12x$

$(x^2 + 12x + 36) - 36$

Using perfect square factoring,

Ans $\boxed{(x+6)^2 - 36}$

Example 2: Complete the square on $x^2 - 20x$.

Solution

$(-\frac{20}{2})^2 = (-10)^2 = 100$

Thus, we have

$x^2 - 20x$

$(x^2 - 20x + 100) - 100$

Using perfect square factoring,

Ans $\boxed{(x-10)^2 - 100}$

Example 3: Complete the square on $x^2 + 16x$.

Solution

$(\frac{16}{2})^2 = 8^2 = 64$

Thus, we have

$x^2 + 16x$

$(x^2 + 16x + 64) - 64$

Using perfect square factoring,

Ans $\boxed{(x+8)^2 - 64}$

E. Completing the Square–Part 2

Sometimes the coefficient of x^2 is **not** 1.

We must factor it out first.

Warning: The factor out front will change the "correction".

F. Examples–Part 2

Example 1: Complete the square on $2x^2 - 36x$.

Solution

$2x^2 - 36x$

$2(x^2 - 18x)$

Now $(\frac{18}{2})^2 = 9^2 = 81$, so we have

$$2(x^2 - 18x + 81) - 162$$

Note that when we added 81, we really added 162,
since the 81 is being multiplied by 2.

Ans $\boxed{2(x-9)^2 - 162}$

Example 2: Complete the square on $3x^2 + 24x$.

Solution

$3x^2 + 24x$

$3(x^2 + 8x)$

Now $(\frac{8}{2})^2 = 4^2 = 16$, so we have

$$3(x^2 + 8x + 16) - 48$$

Note that when we added 16, we really added 48,
since the 16 is being multiplied by 3.

Ans $\boxed{3(x + 4)^2 - 48}$

Example 3: Complete the square on $-2x^2 + 20x$.

Solution

$-2x^2 + 20x$

$-2(x^2 - 10x)$

Now $(-\frac{10}{2})^2 = (-5)^2 = 25$, so we have

$$-2(x^2 - 10x + 25) + 50$$

Note that when we added 25, we really **subtracted** 50, since the 25 is being multiplied by -2. We need to add 50 to cancel the -50.

Ans $\boxed{-2(x - 5)^2 + 50}$

G. Completing the Square–Part 3

If there is a number already present, we just ignore it, and do everything to the first two terms.

H. Examples–Part 3

Example 1: Complete the square on $2x^2 + 24x + 17$.

Solution

$2x^2 + 24x + 17$

$2(x^2 + 12x) + 17$

Now $(\frac{12}{2})^2 = 6^2 = 36$, so we have

$$2(x^2 + 12x + 36) + 17 - 72 \quad \text{(correcting for } 2 \cdot 36\text{)}$$

Ans $\boxed{2(x+6)^2 - 55}$

Example 2: Complete the square on $-3x^2 + 12x - 5$.

Solution

$-3x^2 + 12x - 5$

$-3(x^2 - 4x) - 5$

Now $(-\frac{4}{2})^2 = (-2)^2 = 4$, so we have

$$-3(x^2 - 4x + 4) - 5 + 12 \quad \text{(correcting for } (-3)(4))$$

Ans $\boxed{-3(x-2)^2 + 7}$

I. Completing the Square–Part 4

If you "can't factor" the x^2 coefficient out of the first 2 terms, you will need to use fractions.

J. Examples–Part 4

Example 1: Complete the square on $2x^2 + 3x - 7$.

Solution

$2x^2 + 3x - 7$

$2(x^2 + \_\_\, x) - 7$ need fraction: "introduce 3, kill 2"

$2(x^2 + \frac{3}{2}x) - 7$

Now $\left(\frac{\frac{3}{2}}{2}\right)^2 = (\frac{3}{4})^2 = \frac{9}{16}$, so we have

$$2(x^2 + \tfrac{3}{2}x + \tfrac{9}{16}) - 7 - \tfrac{9}{8} \quad \text{(note that } 2 \cdot \tfrac{9}{16} = \tfrac{9}{8})$$

$$2(x^2 + \tfrac{3}{2}x + \tfrac{9}{16}) - \tfrac{56}{8} - \tfrac{9}{8}$$

Ans $\boxed{2(x + \frac{3}{4})^2 - \frac{65}{8}}$

Example 2: Complete the square on $-3x^2 + 2x + 5$.

Solution

$-3x^2 + 2x + 5$

$-3(x^2 - \_\_\, x) + 5$ need fraction: "introduce 2, kill 3"

$-3(x^2 - \frac{2}{3}x) + 5$

Now $\left(\frac{-\frac{2}{3}}{2}\right)^2 = (-\frac{1}{3})^2 = \frac{1}{9}$, so we have

$-3(x^2 - \frac{2}{3}x + \frac{1}{9}) + 5 + \frac{1}{3}$ (note that $-3(\frac{1}{9}) = -\frac{1}{3}$)

$-3(x^2 - \frac{2}{3}x + \frac{1}{9}) + \frac{15}{3} + \frac{1}{3}$

Ans $\boxed{-3(x - \frac{1}{3})^2 + \frac{16}{3}}$

K. Summary of Method

1. Factor the leading coefficient out of the square and linear term **only**.

2. Inside the quantity, add in the square of half the linear coefficient.

3. Make the "appropriate" correction on the outside.

4. Use perfect square factoring and simplify.

L. Closing Remarks

The purpose of completing the square is to be able to create a squared quantity so that we can use the square root principle on quadratic equations with unfactorable trinomials. We will pursue this in the subsequent sections.