7.2C Absolute Value and Roots

A. Absolute Value Discussion

Recall that absolute value $|\cdot|$ means **distance from the origin**.

We think of absolute value of numbers as "make it positive", but of course that doesn't work for variables. (See Sections 2.3 and 2.7)

Recall that $|-3| = 3$ and $|4| = 4$ etc.

We know consider a **third interpretation**.

Notice the following:

$$|6| = 6$$

$$|3| = 3$$

$$|0| = 0$$

If the number inside is positive or zero, the absolute value does **nothing**.

Thus $|x| = x;$ if $x \geq 0$.

Notice the following:

$$|-3| = 3$$

$$|-4| = 4$$

In this case, the sign changes.

Question: How can we change from -3 to 3 or -4 to 4 **without** using absolute value signs?

Answer: Multiply by -1

Thus notice that:

$|-3|$ is the same as $-(-3)$

$|-4|$ is the same as $-(-4)$

Thus, $|x| = -x;\ \text{if } x < 0$.

B. Definition of Absolute Value: Three Forms

1. For **numbers** only, "make it positive"

2. **True Definition:** distance from the origin

   This is the correct definition, and works for numbers or variables. This is needed for **equations** or **inequalities**

3. **Piecewise Definition:**

$$\boxed{|x| = \begin{cases} x; & \text{if } x \geq 0 \\ -x; & \text{if } x < 0 \end{cases}}$$

C. Comments on the Piecewise Definition

1. The piecewise definition is the "formal definition" in terms of an algebraic formula.

2. The piecewise definition does **not** mean that $|x| = \pm\, x$ or some such nonsense.
There is only **one** answer to $|x|$; however, the answer we choose **depends** on what's inside.

3. When an object has more than one "formula", and the expression you choose depends on some conditions, we say that the object is **piecewise defined**.

4. See MTH103 for more on piecewise definitions.

D. Use of the Piecewise Definition of $|x|$ in Examples

Example 1: Find $|7 - \sqrt{3}|$ exactly

Solution

$7 - \sqrt{3} \geq 0$; so we use $|x| = x$ in **this** case

Thus $|7 - \sqrt{3}| = 7 - \sqrt{3}$

Ans $\boxed{7 - \sqrt{3}}$

Example 2: Find $|1 - \sqrt{17}|$ exactly

Solution

$1 - \sqrt{17} < 0$; so we use $|x| = -x$ in **this** case

Thus $|1 - \sqrt{17}| = -(1 - \sqrt{17})$

Ans $\boxed{-1 + \sqrt{17}}$

Example 3: Find $|\sqrt{3} - \sqrt{10}|$ exactly

Solution

$\sqrt{3} - \sqrt{10} < 0$; so we use $|x| = -x$ in **this** case

Thus $|\sqrt{3} - \sqrt{10}| = -(\sqrt{3} - \sqrt{10})$

Ans $\boxed{-\sqrt{3} + \sqrt{10}}$

E. Roots and Powers

1. $\boxed{(\sqrt[n]{x})^n = x}$

This is by definition of the nth root!

Thus "Root First, Then Power" $\Longrightarrow$ **Cancel!**

$(\sqrt{x})^2 = x$

$(\sqrt[3]{x})^3 = x$, etc.

2. The problem with $\sqrt[n]{x^n}$

We know that this is **not** the same situation.

Recall that we are only allowed to move powers inside if x is not simultaneously negative with n even.

Consider $\sqrt{x^2}$:

$$\sqrt{4^2} = \sqrt{16} = 4$$

$$\sqrt{0^2} = \sqrt{0} = 0$$

$$\sqrt{(-3)^2} = \sqrt{9} = 3 \quad \textbf{not } -3!$$

We see that, in fact, $\sqrt{x^2} = |x|$, since the answer is always positive (or zero)

We have a similar situation for all **even** roots:

$$\sqrt{x^2} = |x|$$

$$\sqrt[4]{x^4} = |x|$$

$$\sqrt[6]{x^6} = |x|$$

Since we don't have any problem with **odd** roots, they just cancel:

$$\sqrt[3]{x^3} = x$$

$$\sqrt[5]{x^5} = x$$

Hence, we get another piecewise definition, depending on whether the index is even or odd:

$$\boxed{\sqrt[n]{x^n} = \begin{cases} x; & \text{if } n \text{ is odd} \\ |x|; & \text{if } n \text{ is even} \end{cases}}$$

Thus "Power First, Then Root" $\Longrightarrow$ cancel only if the index is odd; otherwise absolute value!

F. Examples

Example 1: Find $\sqrt[3]{(7-\sqrt{3})^3}$ exactly

Solution

Since the index is **odd**, we use $\sqrt[n]{x^n} = x$ in **this** case

Thus $\sqrt[3]{(7-\sqrt{3})^3} = 7-\sqrt{3}$

Ans $\boxed{7-\sqrt{3}}$

Example 2: Find $\sqrt[4]{(10-\sqrt{5})^4}$ exactly

Solution

Since the index is **even**, we use $\sqrt[n]{x^n} = |x|$ in **this** case

Thus $\sqrt[4]{(10-\sqrt{5})^4} = |10-\sqrt{5}|$

Now $10-\sqrt{5} \geq 0$; so we use $|x| = x$ in **this** case

Thus $|10-\sqrt{5}| = 10-\sqrt{5}$

Ans $\boxed{10-\sqrt{5}}$

Here's where it gets interesting!

Example 3: Find $\sqrt[6]{(1-\sqrt{7})^6}$ exactly

Solution

Since the index is **even**, we use $\sqrt[n]{x^n} = |x|$ in **this** case

Thus $\sqrt[6]{(1-\sqrt{7})^6} = |1-\sqrt{7}|$

Now $1-\sqrt{7} < 0$; so we use $|x| = -x$ in **this** case

Thus $|1-\sqrt{7}| = -(1-\sqrt{7})$

Ans $\boxed{-1+\sqrt{7}}$

Example 4: Find $\sqrt{(\sqrt[3]{6}-\sqrt[3]{13})^2}$ exactly

Solution

Since the index is **even**, we use $\sqrt[n]{x^n} = |x|$ in **this** case

Thus $\sqrt{(\sqrt[3]{6}-\sqrt[3]{13})^2} = |\sqrt[3]{6}-\sqrt[3]{13}|$

Now $\sqrt[3]{6}-\sqrt[3]{13} < 0$; so we use $|x| = -x$ in **this** case

Thus $|\sqrt[3]{6}-\sqrt[3]{13}| = -(\sqrt[3]{6}-\sqrt[3]{13})$

Ans $\boxed{-\sqrt[3]{6}+\sqrt[3]{13}}$

G. Summary of Formulas

1. $\boxed{\sqrt[n]{x} = x^{\frac{1}{n}}}$ UNLESS index is even with x possibly negative

2. $\boxed{x^{\frac{m}{n}} = (\sqrt[n]{x})^m}$ UNLESS index is even with x possibly negative

3. $\boxed{(\sqrt[n]{x})^m = \sqrt[n]{x^m}}$ UNLESS index is even with x possibly negative

4. Piecewise Definition of $|x|$:

$$\boxed{|x| = \begin{cases} x; & \text{if } x \geq 0 \\ -x; & \text{if } x < 0 \end{cases}}$$

5. $\boxed{(\sqrt[n]{x})^n = x}$ "Root First, Then Power" $\Longrightarrow$ CANCEL

6. Piecewise Definition of $\sqrt[n]{x^n}$:

$$\boxed{\sqrt[n]{x^n} = \begin{cases} x; & \text{if } n \text{ is odd} \\ |x|; & \text{if } n \text{ is even} \end{cases}}$$