7.2B Roots, Radicals, and Rational Exponents

A. Discussion

Consider the symbol $4^{\frac{1}{2}}$.

If we were to multiply it by itself, and the standard rules would still hold, then

$4^{\frac{1}{2}} \cdot 4^{\frac{1}{2}} = 4^{\frac{2}{2}}$ (by adding exponents)

Thus $4^{\frac{1}{2}} \cdot 4^{\frac{1}{2}} = 4^1 = 4$.

Hence, $4^{\frac{1}{2}}$ is the number multiplied by itself to give you 4.

Thus it makes sense to identify $4^{\frac{1}{2}}$ as $\sqrt{4}$.

In general, we **define** $x^{\frac{1}{2}} = \sqrt{x}$

Similarly, we **define** $x^{\frac{1}{n}} = \sqrt[n]{x}$ for the same reasons.

Since we expect that $x^{\frac{m}{n}} = (x^{\frac{1}{n}})^m$, we also **define** $x^{\frac{m}{n}} = (\sqrt[n]{x})^m$

Since an even root can't accept a negative number,
we can't have x negative when the index is even.

B. Conversion Rules

1. $x^{\frac{1}{n}} = \sqrt[n]{x}$

2. $x^{\frac{m}{n}} = (\sqrt[n]{x})^m$ "root first, then power"

Note: These are not valid if $x < 0$ when n is even.

C. Evaluation Examples

Example 1: Find $8^{\frac{1}{3}}$

Solution

$$8^{\frac{1}{3}} = \sqrt[3]{8}$$

Ans $\boxed{2}$

Example 2: Find $27^{\frac{2}{3}}$

Solution

$$27^{\frac{2}{3}} = (\sqrt[3]{27})^2 = 3^2$$

Ans $\boxed{9}$

Example 3: Find $25^{\frac{3}{2}}$

Solution

$$25^{\frac{3}{2}} = (\sqrt{25})^3 = 5^3$$

Ans $\boxed{125}$

Example 4: Find $(-32)^{\frac{2}{5}}$

Solution

$$(-32)^{\frac{2}{5}} = (\sqrt[5]{-32})^2 = (-2)^2$$

Ans $\boxed{4}$

Example 5: Find $(-64)^{\frac{3}{2}}$

Solution

Since the denominator is even and the number is negative, this is not a real number!

Ans $\boxed{\text{not a real number}}$

D. More Evaluation Examples

Recall that negative exponents mean reciprocal, and to do the root of a fraction, you do top and bottom separately.

Example 1: Find $(-8)^{-\frac{2}{3}}$

Solution

$$(-8)^{-\frac{2}{3}} = \frac{1}{(-8)^{\frac{2}{3}}} = \frac{1}{(\sqrt[3]{-8})^2} = \frac{1}{(-2)^2}$$

Ans $\boxed{\frac{1}{4}}$

Example 2: Find $\left(\frac{16}{25}\right)^{-\frac{3}{2}}$

Solution

$$\left(\frac{16}{25}\right)^{-\frac{3}{2}} = \left(\frac{25}{16}\right)^{\frac{3}{2}} = \left(\sqrt{\frac{25}{16}}\right)^{3} = \left(\frac{5}{4}\right)^{3}$$

Ans $\boxed{\frac{125}{64}}$

Example 3: Find $\left(-\frac{27}{8}\right)^{-\frac{2}{3}}$

Solution

$$\left(-\frac{27}{8}\right)^{-\frac{2}{3}} = \left(-\frac{8}{27}\right)^{\frac{2}{3}} = \left(\sqrt[3]{-\frac{8}{27}}\right)^{2} = \left(-\frac{2}{3}\right)^{2}$$

Ans $\boxed{\frac{4}{9}}$

E. Radicals and Moving Powers

A root with variables inside is called a **radical**.

We know that we have the relationship $x^{\frac{m}{n}} = (\sqrt[n]{x})^m$.

A question arises: When simplifying radicals using the above relationship, can we move the power to the inside of the root?

1. If the index is **odd**, the answer is **yes**.

2. If the index is **even**, we can do it **only if the variables are not allowed to be negative**.

In particular, $(\sqrt[n]{x})^m \neq \sqrt[n]{x^m}$ for even index n (in general).

F. Converting Variables with Fractional Exponents to Radical Form

We use the relationship $x^{\frac{m}{n}} = (\sqrt[n]{x})^m$ "root first, then power".

If we don't have the **bad situation** of "even index, with variables possibly negative", we can simplify further by bringing the root to the inside.

Example 1: Convert $(2x^2y)^{\frac{2}{5}}$ to radical form

Solution

$(2x^2y)^{\frac{2}{5}} = (\sqrt[5]{2x^2y})^2$

Since the index is odd, we can bring the power to the inside: $\sqrt[5]{(2x^2y)^2}$

Ans $\boxed{\sqrt[5]{4x^4y^2}}$

Example 2: Convert $(3xy^2)^{\frac{3}{4}}$ to radical form. Assume variables are **not** negative.

Solution

$$(3xy^2)^{\frac{3}{4}} = (\sqrt[4]{3xy^2})^3$$

The index is even, but the variables are assumed to not be negative, so we can bring the power to the inside.

$$\sqrt[4]{(3xy^2)^3}$$

Ans $\boxed{\sqrt[4]{27x^3y^6}}$

Note: If the extra condition "assume variables are not negative" was not present, then we would not be able to do anything! Neither the conversion in the first step, nor moving the power to the inside would be legitimate!

G. Closing Comments

This issue of even index paired with variables possibly being negative is a serious problem. We will address this issue in the next section, and we will see that certain expressions can simplify strangely as a result.