6.1B Simplifying/Multiplying/Dividing Rational Expressions

A. Simplifying Rational Expressions

To simplify a fraction, we divide top and bottom by common factors. To do this, we have to have the top and bottom factored!

Example 1: Simplify $\dfrac{x^2-4x}{xy-4y}$

Solution

Factor the top and bottom: $\dfrac{x(x-4)}{y(x-4)}$

Divide top and bottom by the common factor $x-4$:

Ans $\boxed{\dfrac{x}{y}}$

Example 2: Simplify $\dfrac{2x^2+x-3}{8x^3+27}$

Solution

Factor top and bottom:

Top:

$$\begin{array}{c|ll} 2x^2+x-3 & \boxed{-6} & \text{TSP: } +, - \\ \hline 2x^2+2x-x-3 & -2 & \\ 2x^2+3x-2x-3 & -6 \; \surd & \\ x(2x+3)-1(2x+3) & & \\ (2x+3)(x-1) & & \end{array}$$

Bottom:

$$8x^3 + 27$$

$$(2x)^3 + 3^3$$

$$(2x+3)(4x^2 - 6x + 9)$$

Thus we have $\dfrac{2x^2 + x - 3}{8x^3 + 27} = \dfrac{(2x+3)(x-1)}{(2x+3)(4x^2-6x+9)}$

Dividing top and bottom by the common factor $2x + 3$:

Ans $\boxed{\dfrac{x-1}{4x^2-6x+9}}$

B. Comments on Canceling Common Factors

We think of dividing the top and bottom by a factor as "canceling".

We have to be careful when canceling and understand when it is allowed.

1. To be able to cancel something, it must be a factor. This means that it is separated off and multiplying the rest of the expression.

2. As a general rule, we think of $+$ or $-$ signs like "glue". If we see a $+$ or $-$ sign next to an object we'd like to cancel, we can't do it!

3. If you can't cancel something, you may be able to factor. This "unhooks" the object.

Canceling Examples for Discussion

$$\frac{x^2 - 6}{x - 2}$$

Here we **can't** do any canceling. x^2 is attached like glue to -6 on the top, and x is attached like glue to -2 on the bottom.

$$\frac{x(x - 6)}{x(x - 2)}$$

Here we **can** cancel the "left x's ", because there are no $+$ or $-$ signs next to it. It is unhooked. However, the x's on the inside of the parentheses on the top and bottom **can't** be canceled; they are attached by -6 and -2 respectively. Also, you **can't** cancel the -6 and -2 for the same reason; they are attached to the x.

$$\frac{x + 3}{x + 3 + y}$$

You **can't** cancel the $x + 3$, it is attached on the bottom by a $+$ sign.

$$\frac{x + 3}{(x + 3)y}$$

Here you **can** cancel the $x + 3$. Thinking of $x + 3$ as one object, there is no $+$ or $-$ sign next to it in either the top or the bottom. It is unhooked.

$$\frac{x^2+3x}{x^2+4x}$$

Here you **can't** do any canceling initially, as everything is attached by $+$ or $-$ signs. However, if you factor the top and bottom: $\frac{x(x+3)}{x(x+4)}$, then you "unhook" an x that **can** be canceled.

BAD CANCELING is a common error. Make sure you understand when you can and when you can't cancel!

C. Multiplying Rational Expressions

To multiply fractions, we multiply numerators and denominators. However, we factor everything first, so we may cancel common factors.

Note: You can cancel vertically and diagonally, but never horizontally.

Example 1: Multiply $\dfrac{x^2-9}{x^3+8}\cdot\dfrac{x^2-x-6}{x^3-27}$

Solution

Factor first:

$$\frac{(x+3)(x-3)}{(x+2)(x^2-2x+4)}\cdot\frac{(x+2)(x-3)}{(x-3)(x^2+3x+9)}$$

Cancel common factors; "diagonal cancellation"

$$\frac{(x+3)\cancel{(x-3)}}{\cancel{(x+2)}(x^2-2x+4)}\cdot\frac{\cancel{(x+2)}(x-3)}{\cancel{(x-3)}(x^2+3x+9)}$$

Leaving the answer in factored form:

Ans $\boxed{\dfrac{(x+3)(x-3)}{(x^2-2x+4)(x^2+3x+9)}}$

Example 2: Multiply $\dfrac{2x^2+5xy+2y^2}{4x^2-y^2}\cdot\dfrac{2x^2+xy-y^2}{x^2+xy-2y^2}$

Solution

We have four factoring problems: $\dfrac{\boxed{1}2x^2+5xy+2y^2}{\boxed{2}4x^2-y^2}\cdot\dfrac{\boxed{3}2x^2+xy-y^2}{\boxed{4}x^2+xy-2y^2}$

Factor $\boxed{1}$:

$$\begin{array}{c|ll}
2x^2+5xy+2y^2 & \boxed{4} & \text{TSP: } +,+ \\
\hline
2x^2+xy+4xy+2y^2 & 4\ \surd & \\
x(2x+y)+2y(2x+y) & & \\
(2x+y)(x+2y) & &
\end{array}$$

Factor $\boxed{2}$:

$$4x^2-y^2=(2x)^2-y^2=(2x+y)(2x-y)$$

Factor $\boxed{3}$:

$$\begin{array}{c|ll}
2x^2+xy-y^2 & \boxed{-2} & \text{TSP: } +,- \\
\hline
2x^2+2xy-xy-y^2 & -2\ \surd & \\
2x(x+y)-y(x+y) & & \\
(x+y)(2x-y) & &
\end{array}$$

Factor $\boxed{4}$:

$$\begin{array}{c|ll}
x^2+xy-2y^2 & \boxed{-2} & \text{TSP: } +,- \\
\hline
x^2+2xy-xy-2y^2 & -2\ \surd & \\
x(x+2y)-y(x+2y) & & \\
(x+2y)(x-y) & &
\end{array}$$

Now put it all together:

$$\frac{2x^2+5xy+2y^2}{4x^2-y^2}\cdot\frac{2x^2+xy-y^2}{x^2+xy-2y^2}=\frac{(2x+y)(x+2y)}{(2x+y)(2x-y)}\cdot\frac{(x+y)(2x-y)}{(x+2y)(x-y)}$$

Canceling common factors:

$$\frac{\cancel{(2x+y)}\cancel{(x+2y)}}{\cancel{(2x+y)}\cancel{(2x-y)}}\cdot\frac{(x+y)\cancel{(2x-y)}}{\cancel{(x+2y)}(x-y)}$$

Ans $\boxed{\dfrac{x+y}{x-y}}$

D. Dividing Rational Expressions

To divide fractions, we invert the second fraction and multiply.

Example 1: Divide $\dfrac{8x^3+27}{64x^3-1}\div\dfrac{4x^2-9}{16x^2+4x+1}$

Solution

Multiply $\dfrac{8x^3+27}{64x^3-1}\cdot\dfrac{16x^2+4x+1}{4x^2-9}$

We have four factoring problems: $\dfrac{\boxed{1}\,8x^3+27}{\boxed{2}\,64x^3-1}\cdot\dfrac{\boxed{3}\,16x^2+4x+1}{\boxed{4}\,4x^2-9}$

Factor $\boxed{1}$:

$$8x^3+27=(2x)^3+3^3=(2x+3)(4x^2-6x+9)$$

Factor $\boxed{2}$:

$$64x^3-1=(4x)^3-1^3=(4x-1)(16x^2+4x+1)$$

Factor $\boxed{3}$: $16x^2 + 4x + 1$ is prime (since prime factor of $\boxed{2}$)

Factor $\boxed{4}$:

$$4x^2 - 9 = (2x)^2 - 3^2 = (2x+3)(2x-3)$$

Now put it all together:

$$\frac{(2x+3)(4x^2-6x+9)}{(4x-1)(16x^2+4x+1)} \cdot \frac{16x^2+4x+1}{(2x+3)(2x-3)}$$

Now cancel common factors:

$$\frac{\cancel{(2x+3)}(4x^2-6x+9)}{(4x-1)\cancel{(16x^2+4x+1)}} \cdot \frac{\cancel{16x^2+4x+1}}{\cancel{(2x+3)}(2x-3)}$$

Ans $\boxed{\dfrac{4x^2-6x+9}{(4x-1)(2x-3)}}$