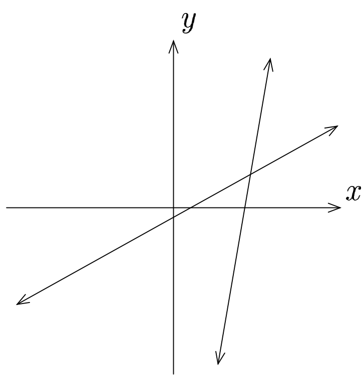
4.1 Systems of Linear Equations

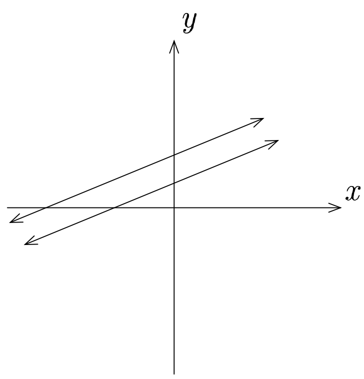
A. Introduction

Suppose we have **two** lines.

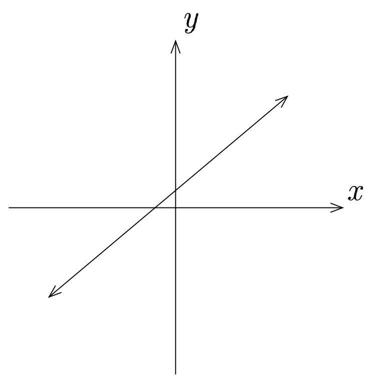
We have three possibilities:



the lines intersect



the lines are parallel



the lines overlap (coincide)

Thus, if we have a **system of linear equations**;
that is, "two line equations in x and y", then . . .

If the

1. lines intersect: we have **one** ordered pair that lies on both lines

2. lines are parallel: **no** ordered pairs lie on both lines

3. lines coincide: **infinitely many** ordered pairs lie on both lines

Ordered pairs (x, y) that work in **both** equations are called **solutions** to the system of equations. They represent the intersection points of the two lines. Thus a system has **one solution**, **no solutions**, or **infinitely many solutions**.

B. Checking Solutions to Systems

If an ordered pair is a solution, it must work in both equations.
We plug the trial point into each system.
It is a solution **only** if it works in **both**.

Example: Check to see if $(3, -1)$ is a solution to the system $\begin{cases} 5x + y = 14 \\ x + 2y = -8 \end{cases}$.

Solution

Plug $(3, -1)$ into each equation and see if the equation is true.

$$5(3) + (-1) \stackrel{?}{=} 14 \implies 15 - 1 \stackrel{?}{=} 14 \quad \checkmark$$

$$3 + 2(-1) \stackrel{?}{=} -8 \implies 3 - 2 \stackrel{?}{=} -8 \quad \text{X}$$

Doesn't work in both!

Ans $\boxed{\text{not a solution}}$

C. Solving a System By Substitution

1. **Pick** one of the two equations (your choice).
2. Solve the equation for one of the two variables (your choice).
3. Substitute into the **other** equation with that variable.
4. After solving, the answer is one of the coordinates.
5. Substitute back into any equation to get the other variable answer.

Example: Solve $\begin{cases} 3x + 8y &= 19 \\ 4x + y &= 6 \end{cases}$ for (x, y) using substitution.

Solution

Pick equation 2 and solve for y (your choice):

$$4x + y = 6 \implies y = 6 - 4x$$

Now replace y with $6 - 4x$ in equation 1:

$$3x + 8(6 - 4x) = 19$$

$$3x + 48 - 32x = 19$$

$$-29x + 48 = 19$$

$$-29x = -29$$

$$x = 1$$

Now $y = 6 - 4x$, so $y = 6 - 4(1) = 6 - 4 = 2$.

Thus $(x, y) = (1, 2)$.

Ans $\boxed{(1, 2)}$

Note: We picked equation 2 and picked solving for y because the variable y didn't have a coefficient in front of it. This makes things simpler. You can pick a different equation and different variable when you start, but the work will just be more involved because of fractions. For example:

If we had picked equation 1 and solved for x:

$$3x + 8y = 19 \implies 3x = 19 - 8y \implies x = \frac{19 - 8y}{3}$$

Replacing x with $\dfrac{19 - 8y}{3}$ in equation 2:

$$4\left(\frac{19 - 8y}{3}\right) + y = 6$$

$$\frac{76 - 32y}{3} + y = 6$$

$$3\left(\frac{76 - 32y}{3} + y\right) = 3 \cdot 6$$

$$76 - 32y + 3y = 18$$

$$76 - 29y = 18$$

$$-29y = -58$$

$$y = 2$$

Then $x = \dfrac{19 - 8y}{3}$, so $x = \dfrac{19 - 8(2)}{3} = \dfrac{19 - 16}{3} = \dfrac{3}{3} = 1.$

Thus we get $(1, 2)$ as before.

D. Comments on Solving Systems

1. If when solving a system, you get a **false** statement (like $0 = 3$),
   the system has **no solutions**.

2. If when solving a system, you get a **true** statement (like $2 = 2$),
   the system has **an infinite number of solutions**
   (namely all points on the common line).

The above situations happen when the variables "disappear" during the solution process.

E. Solving a System By Elimination

Another way to solve a system (sometimes easier) is to **eliminate** one of the variables.

1. Get coefficients opposite in sign of the variable you want to eliminate.
   Do this by multiplying each equation by a number.

2. Add equations to eliminate the variable.

3. Solve the equation to get one coordinate.

4. Substitute back into one of the equations to get the other coordinate.

Example 1: Solve the system $\begin{cases} x - y &= 4 \\ x + y &= 10 \end{cases}$ for (x, y) by elimination.

Solution

Notice that by adding both equations the variable y disappears: $2x = 14$.

Then $x = 7$.

Plugging into the second equation, say, we get $7 + y = 10$, so $y = 3$.

Thus, we have $(x, y) = (7, 3)$

Ans $\boxed{(7, 3)}$

Example 2: Solve the system $\begin{cases} 5x + y = 14 \\ x + 2y = -8 \end{cases}$ for (x, y) by elimination.

Solution

We decide to eliminate y (arbitrary choice).
To cancel, we must multiply by constants . . .

Multiply equation 1 by -2:

$$\begin{cases} -10x - 2y = -28 \\ x + 2y = -8 \end{cases}$$

Now add the equations: $-9x = -36 \implies x = 4$.

Plug $x = 4$ into equation 1 (your choice):

$$5(4) + y = 14 \implies 20 + y = 14 \implies y = -6$$

Thus $(x, y) = (4, -6)$.

Ans $\boxed{(4, -6)}$

Example 3: Solve the system $\begin{cases} 2x + 3y = -4 \\ 3x - 2y = 7 \end{cases}$ for (x, y) by elimination.

Solution

We decide to eliminate x (arbitrary choice).
To cancel, we must multiply by constants . . .

Multiply equation 1 by 3 and equation 2 by -2:

$$\begin{cases} 6x + 9y = -12 \\ -6x + 4y = -14 \end{cases}$$

Now add the equations: $13y = -26 \implies y = -2$.

Plug $y = -2$ into equation 2 (your choice):

$$3x - 2(-2) = 7 \implies 3x + 4 = 7 \implies 3x = 3 \implies x = 1$$

Thus $(x, y) = (1, -2)$.

Ans $\boxed{(1, -2)}$

Example 4: Solve the system $\begin{cases} 3x - 2y = 8 \\ -6x + 4y = -16 \end{cases}$ for (x, y) by elimination.

Solution

We decide to eliminate x (arbitrary choice).
To cancel, we must multiply by constants . . .

Multiply equation 1 by 2:

$$\begin{cases} 6x - 4y = 16 \\ -6x + 4y = -16 \end{cases}$$

Now add the equations: $0 = 0$

The variables disappeared and we got a true statement!

Thus we have infinitely many solutions.

Ans infinitely many solutions! all points on the line $3x - 2y = 8$

F. Closing Comments

1. If a system of equations has no solution, we sometimes say that the system is **inconsistent**.

2. If a system of equations has infinitely many solutions, we say that the equations are **dependent**.

3. Another solution method is called **solution by graphing**.
   Here each line is graphed, and the intersection point is determined by picture.
   However, this is usually difficult to get an accurate answer.