1.4A Properties of Exponents I

A. Product Rule

Notice the following: $4^2 \cdot 4^3 = (4 \cdot 4) \cdot (4 \cdot 4 \cdot 4) = 4^5 = 4^{2+3}$

Similarly, $x^2 \cdot x^3 = x^5 = x^{2+3}$

In general, $x^m \cdot x^n = x^{m+n}$

Thus, when we multiply powers, we **add exponents**.

Example:

Find $x^7 \cdot x^5$

$x^7 \cdot x^5 = \boxed{x^{12}}$ (by adding exponents)

B. Quotient Rule

Notice the following: $\frac{4^5}{4^2} = \frac{4 \cdot 4 \cdot 4 \cdot 4 \cdot 4}{4 \cdot 4} = \frac{4 \cdot 4 \cdot 4 \cdot \cancel{4} \cdot \cancel{4}}{\cancel{4} \cdot \cancel{4}} = 4 \cdot 4 \cdot 4 = 4^3 = 4^{5-2}$

Similarly, $\frac{x^5}{x^2} = x^3 = x^{5-2}$

In general, $\frac{x^m}{x^n} = x^{m-n}$

Thus, when we divide powers, we **subtract exponents**.

Example:

Find $\frac{x^{11}}{x^6}$

$\frac{x^{11}}{x^6} = \boxed{x^5}$ (by subtracting exponents)

C. Power Rule

Notice the following: $(4^2)^3 = 4^2 \cdot 4^2 \cdot 4^2 = 4^{2+2+2} = 4^6 = 4^{2 \cdot 3}$

Similarly, $(x^2)^3 = x^6 = x^{2 \cdot 3}$

In general, $(x^m)^n = x^{mn}$

Thus, when we take a power of a power, we **multiply exponents**.

Example:

Find $(x^5)^6$

$(x^5)^6 = \boxed{x^{30}}$ (by multiplying exponents)

WARNING: Don't get the product and power rules confused:

$\boxed{x^4 x^3 = x^7 \text{ but } (x^4)^3 = x^{12}}$

D. Zero Power

Notice the following: $\frac{4^8}{4^8} = \frac{\cancel{4^8}}{\cancel{4^8}} = 1$, but by the quotient rule, $\frac{4^8}{4^8} = 4^0$.

Thus, we see that $4^0 = 1$

Similarly, $123^0 = 1$.

The above argument works for any number, except zero, because we would have $\frac{0^8}{0^8} = \frac{0}{0}$ which is indeterminate.

In general, $x^0 = 1$, if $x \neq 0$

E. Negative Exponents

Notice the following: $\frac{4^2}{4^5} = \frac{4 \cdot 4}{4 \cdot 4 \cdot 4 \cdot 4 \cdot 4} = \frac{\cancel{4} \cdot \cancel{4}}{4 \cdot 4 \cdot 4 \cdot \cancel{4} \cdot \cancel{4}} = \frac{1}{4 \cdot 4 \cdot 4} = \frac{1}{4^3}$.

However $\frac{4^2}{4^5} = 4^{-3}$ by the quotient rule.

Thus, $4^{-3} = \frac{1}{4^3}$.

Similarly, $x^{-3} = \frac{1}{x^3}$.

In general, $x^{-m} = \frac{1}{x^m}$.

Thus, a negative exponent means **reciprocal**.

Example:

Find 4^{-3}

$$4^{-3} = \frac{1}{4^3} = \boxed{\frac{1}{64}} \text{ (negative exponent rule)}$$

F. Switch Rule

By using the reciprocal idea for negative exponents, we get the switch rule:

$$\frac{x^{-m}}{y^{-n}} = \frac{y^n}{x^m}$$

Thus, in a fraction, powers in the top get sent to the bottom by negative exponents, and powers in the bottom get sent to the top by negative exponents.

Example:

Rewrite without negative exponents: $\frac{x^3y^{-4}}{z^{-5}}$

$$\frac{x^3y^{-4}}{z^{-5}} = \boxed{\frac{x^3z^5}{y^4}}$$ (using the switch rule)

G. Multiple Power Rule

By using the power rule idea, we see that if we have a fraction to a power, we hit every entry with the power (and multiply exponents).

Thus, $\left(\frac{xy}{z}\right)^m = \frac{x^my^m}{z^m}$.

Example:

Find $\left(\frac{x^3y^2}{3z^4}\right)^3$

$$\left(\frac{x^3y^2}{3z^4}\right)^3 = \frac{x^9y^6}{3^3z^{12}} = \boxed{\frac{x^9y^6}{27z^{12}}}$$

H. Comments on Rules

1. It is important to memorize these rules, and to not get them confused.

2. Parentheses are important!:

   $5x^0 = 5 \cdot 1 = 5, \quad \text{but} \quad (5x)^0 = 5^0 x^0 = 1 \cdot 1 = 1$

3. If everything "leaves" a numerator or denominator, you leave "1" behind:

   For example: $\frac{x^{-3}y^{-4}}{z^5} = \frac{1}{x^3y^4z^5}$

4. With fractions, variables not in a fraction are considered to be in the numerator.

   For example: $\frac{2}{3}x^3y^2$ is the same as $\frac{2x^3y^2}{3}$

5. **When using the quotient rule**:

   If you use it on a variable, the answer goes in the numerator. If you don't use it, the variable stays put.

   For example: $\frac{x^4y^5z^6}{x^7y^2z^8w^3} = \frac{x^{-3}y^3z^{-2}}{w^3}$

   In this example, x, y, z are put in the numerator, and w stays put.

6. **When using the multiple power rule**:

   It is only correct to use it if the variables are not added or subtracted:

   $(xy)^2 = x^2y^2$ but $(x+y)^2$ can't be simplified by this rule.

   We will explain how to simplify $(x+y)^2$ later in the course. For the time being, remember the rule:

   DON'T APPLY POWERS ACROSS PLUS OR MINUS SIGNS